80. On the Class Number of Imaginary Quadratic Number Fields

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(Comm. by Zyoiti SUETUNA, M.J.A., June 12, 1964)

1. It was proved by Nagel [3] that there exist infinitely many imaginary quadratic number fields each with class number divisible by a given integer. This fact was also proved by Humbert [2] and Ankeny-Chowla [1] independently.[1]

Let n be a given integer greater than 1 and S a set of a finite number of rational primes fixed once for all. In this note we shall prove, by a method analogous to that used in [1] or [2], the following

THEOREM 1. *There exist infinitely many imaginary quadratic number field F's each with the following two properties:*

i) *the class number of F is a multiple of n,*

ii) *if $S \ni p$, then p is ramified in F.*

2. Let m be a square-free negative integer and d be the discriminant of the imaginary quadratic number field $F=Q(\sqrt{m})$. We denote by k the norm of a primitive[2] ambiguous integral ideal of F, which is different from the principal ideal $(\sqrt{m})$. Thus k is equal to 1 or a positive proper square-free divisor of d different from $-m$. We define now the number q as follows. If n is odd, or n is even and $k=1$, q is the smallest prime factor of n. If n is even, not a power of 2, and $k\neq 1$, then q is the half of the smallest odd prime factor of n. Finally if n is a power of 2 and $k\neq 1$, then q is an arbitrary real number greater than one. In any case we have $n>n/q$.

THEOREM 2. *Case* i) *Let* $m\equiv 1 \pmod 4$. *If m is expressible in the form*

$$m=(kb)^2-4\,ka^n, \tag{1}$$

where a (>1) and b (>0) are integers such that

$$-m>4\,ka^{n/q} \quad (\textit{or equivalently } 4\,ka^n-4\,ka^{n/q}>(kb)^2), \tag{2}$$

then the class number of $F=Q(\sqrt{m})$ is a multiple of n.

Case ii) *Let* $m\equiv 2, 3 \pmod 4$. *If m is expressible in the form*

$$m=(kb)^2-ka^n,$$

where a (>2) and b (>0) are integers such that

$$-m>ka^{n/q} \quad (\textit{or equivalently } ka^n-ka^{n/q}>(kb)^2)$$

and a is odd, then the clas number of $F=Q(\sqrt{m})$ is a multiple of n.

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1) The author wishes to express his hearty thanks to Prof. Leopoldt who has kindly drawn the author's attention to the papers cited in this note.

2) "Primitive" means here "not divisible by a rational integer".