103. On Wiener Homeomorphism between Riemann Surfaces

By Mitsuru NAKAI

Mathematical Institute, Nagoya University

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1. Definition of Wiener homeomorphism (W.H.). In the theory of ideal boundaries of Riemann surfaces, the family of *Wiener functions* ([3], pp. 54–65) and that of *Dirichlet functions* ([3], pp. 65–85) are two main important classes of functions on Riemann surfaces. Let T be a homeomorphism of a Riemann surface R_1 onto another R_2. It is known ([4], [5]) that T is a general quasiconformal homeomorphism (which we shall abbreviate as Q.H.) of R_1 onto R_2 if and only if T preserves bounded continuous Dirichlet functions. In contrast with this, it is natural and has some interest to introduce a class of homeomorphisms between Riemann surfaces preserving bounded continuous Wiener functions. Let $\mathcal{W}(R)$ be the totality of *bounded continuous Wiener functions* on a Riemann surface R.

Definition. *A homeomorphism T of a Riemann surface R_1 onto another R_2 is called a Wiener homeomorphism (which we abbreviate as W.H.) of R_1 onto R_2 if $f\circ T$ belongs to $\mathcal{W}(R_1)$ when and only when f belongs to $\mathcal{W}(R_2)$.*

2. Algebraic and topological criterion of existence of W.H. Let R^* be the *Wiener compactification* ([3], pp. 96–109) of a Riemann surface R and $C(R^*)$ be the totality of real-valued bounded continuous functions on R^*. By definition, any function in $\mathcal{W}(R)$ can be continuously extended to R^* uniquely and so we may consider that $\mathcal{W}(R)\subset C(R^*)$. Since $\mathcal{W}(R)$ is a vector subspace of $C(R^*)$ which is closed under max and min operations ([3], p. 56) and $\mathcal{W}(R)$ separates points in R^* ([3], p. 98), by Stone's theorem ([3], p. 5), $\mathcal{W}(R)$ is dense in $C(R^*)$ with respect to the uniform convergence topology. Hence $\mathcal{W}(R)=C(R^*)$, since $\mathcal{W}(R)$ is uniformly closed. We call $\mathcal{W}(R)$ *Wiener algebra* on R in contrast with Royden algebra ([5]).

Theorem 1. *Any W.H. T of R_1 onto R_2 induces (and is induced by) an algebraic isomorphism $f\to f^\sigma$ of $\mathcal{W}(R_1)$ onto $\mathcal{W}(R_2)$ satisfying $f^\sigma=f\circ T^{-1}$.*[1]

Proof. We have only to show that any algebraic isomorphism $f\to f^\sigma$ of $\mathcal{W}(R_1)$ onto $\mathcal{W}(R_2)$ is induced by a W.H. T of R_1 onto R_2 with $f^\sigma=f\circ T^{-1}$. Since $\mathcal{W}(R_i)=C(R_i^*)$ and R_i^* is compact, any algebraic homomorphism of $\mathcal{W}(R_i)$ onto real numbers is of the form $f\to f(p)$, where p is a unique fixed point in R_i^* determined by this homomorphism. Let $p\in R_1^*$. Then $f\to f^{\sigma^{-1}}(p)$ is an algebraic homo-