36. On Closures of Vector Subspaces. II

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(Comm. by Kinjirô KUNUGI, M.J.A., Feb. 12, 1965)

5. We shall prove in this section the following theorem.[1]

THEOREM 6. *Let M be an infinite dimensional vector subspace of a vector space E, and let τ_0 be a locally convex Hausdorff topology on M. Let us denote by M' the dual of M for the topology τ_0, and by* $\operatorname{codim}(M')$ *the codimension of M' in M^*.*

1° *If* $\operatorname{codim}(M)$ *is infinite, then* $\operatorname{codim}(M)\leqq 2^{\operatorname{codim}(M')}$ *implies that for every projection p of E onto M, there exists a locally convex Hausdorff topology τ on E such that M is dense in E for the topology τ and p is continuous for the topologies τ and τ_0.*

If $\operatorname{codim}(M)$ *is finite, then* $\operatorname{codim}(M)\leqq\operatorname{codim}(M')$ *implies the same conclusion.*

Conversely

2° *If there exists a locally convex Hausdorff topology τ on E such that M is dense in E for the topology τ and a projection p of E onto M is continuous for the topologies τ and τ_0, then either* $\operatorname{codim}(M)\leqq 2^{\operatorname{codim}(M')}$ *or* $\operatorname{codim}(M)\leqq\operatorname{codim}(M')$ *according as* $\operatorname{codim}(M)$ *is infinite or finite.*

Proof of 1°. Suppose first that the dimension of the vector subspace $N=p^{-1}(0)$ is infinite. The inequality $\dim(N)\leqq 2^{\operatorname{codim}(M')}$ shows that there exists a vector subspace N' of N^* such that $\dim(N')\leqq\operatorname{codim}(M')$ and the dual system (N, N') is separated.[2] Let $B_{N'}$ be a base of N'; then, since $\dim(N')\leqq\operatorname{codim}(M')$, we can find a linearly independent subset B of an algebraic supplement of M' in M^* with cardinal number $\dim(N')$. Let φ be a one-to-one mapping of $B_{N'}$ onto B. We define, for each $y'\in B_{N'}$, a linear functional $\bar{y}'$ on E by setting

$$\langle x, \bar{y}'\rangle=\begin{cases}\langle x, \varphi(y')\rangle & \text{for } x\in M,\\ \langle x, y'\rangle & \text{for } x\in N.\end{cases}$$

1) This is a generalization of Theorem 1 of S. Kasahara: Locally convex metrizable topologies which make a given vector subspace dense. Proc. Japan Acad., **40**, 718-722 (1964); to this paper, corrections should be made as follows: Page 718, 'arized' should read 'arisen', and page 719, 'powder' should read 'power'.

2) See Lemma 4 of S. Kasahara: On closures of vector subspaces, I. Proc. Japan Acad., **40**, 723-727 (1964); the preceding sentence of Lemma 4 which begins with the word 'Consequently' should read as follows: Consequently, if the dual system (E, E') is separated, we have $\dim(E)\leqq\cdots$.