62. *On Linear Holonomy Group of Riemannian Symmetric Spaces*

By Jun NAGASAWA
Kumamoto University

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Let M be a connected Riemannian manifold with Riemannian structure g, of dimension n and of class C^∞, and let M_p be the tangent space of M at p. We denote by L_p the group of all linear transformations of M_p. Let A_p be the subgroup of L_p consisting of all elements of L_p which leave invariant the scalar product $g_p(X, Y)$, the curvature tensor R and its successive covariant differentials $\nabla^k R, (k=1, 2, \cdots)$ at p. A_p is a Lie group as a closed subgroup of the Lie group L_p. We denote by $h(p)$ the linear holonomy group of M at p. $h(p)$ is a Lie group, and it's identity component $h(p)^0$ is the restricted linear holonomy group of M at p [3]. In this note we shall denote by G^0 the identity component of a Lie group G.

Theorem 1. *Let M be a Riemannian locally symmetric space, then the restricted holonomy group $h(p)^0$ is contained in A_p^0 at each point p in M.*

Proof. Since M is an analytic Riemannian manifold, the Lie algebra of $h(p)$ consists of the following matrix [3],

$$\sum_{r\ s} \lambda_{rs} R_{rs} \quad \text{where } (R_{rs})_{ij} = (R_{ijrs})_p \,.$$

We take a local coordinate system $(x_1, \cdots, x_n)$ at p such that $\{(\partial/\partial x_1)_p, \cdots, (\partial/\partial x_n)_p\}$ is an orthonormal base of M_p. We express each element of A_p by a matrix with respect to the above base. Then A_p consists of all orthogonal matrices $\|a_{ij}\|$ which satisfy

$$\sum_{\alpha\ \beta\ \gamma\ \delta} a_{i\alpha} a_{j\beta} a_{k\gamma} a_{l\delta} (R_{\alpha\beta\gamma\delta})_p = (R_{ijkl})_p \,.$$

Therefore the Lie algebra of A_p consists of all skew symmetric matrices $\|\mu_{ij}\|$ which satisfy

$$\{\mu_{ih}(R_{hjkl})_p + \mu_{jh}(R_{ihkl})_p + \mu_{kh}(R_{ijhl})_p + \mu_{lh}(R_{ijkh})_p\} = 0.$$

From the Ricci identity we have

$$\nabla_s \nabla_r R_{ijkl} - \nabla_r \nabla_s R_{ijkl} = - \sum_h \{R^h_{irs} R_{hjkl} + R^h_{jrs} R_{ihkl} + R^h_{krs} R_{ijhl} + R^h_{lrs} R_{ijkl}\} \,.$$

Since M is locally symmetric, the left sides of this expression vanish. By lowering the index h and making use of the identities $R_{ijrs} = - R_{ijrs}$,

$$\sum_h \{(R_{ihrs})_p (R_{hjkl})_p + (R_{jkrs})_p (R_{ihkl})_p + (R_{khrs})_p (R_{ijhl})_p + (R_{lhrs})_p (R_{ijkh})_p\} = 0 \,.$$

This means that the Lie algebra af $h(p)$ is contained in the Lie