81. On a Certain Functional-Differential Equation

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1. Let $\mathfrak{M}$ be a family of functions continuous in I: $0 \leqq t < \infty$ in the n-dimensional vector space. Then, we define an operator T satisfying the following conditions:

(i) for any x in $\mathfrak{M}$, Tx is also contained in $\mathfrak{M}$;

(ii) for any sequence $\{x_m\}$ ($x_m \in \mathfrak{M}$) uniformly convergent in I, $\{Tx_m\}$ is also uniformly convergent in I;[1]

(iii) for any scalar functions u and v continuous in I, if $u \leqq v$ is satisfied for $0 \leqq t < s$, where s is an arbitrary constant, then the inequality $Tu \leqq Tv$ remains valid for $t = s$.

Then, let us consider a functional-differential equation such that

$$x' = f(t, x, Tx), \quad x(0) = x_0, \quad t \in I. \tag{1}$$

If we choose the operator and the function f suitably, the equation (1) yields various types of equations, for example, differential equations, integro-differential equations, difference-differential equations, and so on.

In the sequel, the existence of continuous solutions of (1) in I is supposed to be established. However, we need not assume the uniqueness of solutions, so far as we are concerned with the boundedness and stability problems.[2]

2. We first introduce a V-function as follows. Let $V(t, x)$ be a function of t and x satisfying the following conditions:

(i) $V(t, x)$ is continuous and non-negative in I and $|x| < \infty$;

(ii) $V(t, x)$ satisfies the Lipschitz condition such that

$$|V(t, x) - V(t, y)| \leqq k(t)|x - y|,$$

where $k(t)$ is continuous in I;

(iii) $\lim_{|x| \to \infty} V(t, x) = \infty$ uniformly in $t \in I$.

In order to derive some results on the boundedness, it is usefull to introduce two quantities $\mathfrak{d}V(t, x, y)$ and $DV(t, x)$ by setting

$$\mathfrak{d}V(t, x, y) = \overline{\lim_{h \to 0}} \frac{1}{h}(V(t+h, x+hf(t, x, y)) - V(t, x)),$$

$$DV(t, z(t)) = \overline{\lim_{h \to 0}} \frac{1}{h}(V(t+h, z(t+h)) - V(t, z(t)),$$

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1) This means that the operator T is continuous.

2) The author's paper, in which some theorems on the existence and uniqueness of continuous solutions has been discussed, will shortly appear.