111. *On the Rate of Growth of Blaschke Products in the Unit Circle*

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1. Introduction. Let us put

$$B(z)=\prod_{n=1}^{+\infty} b(z, a_n),$$

where $b(z, a)=|a|/a\cdot(a-z)/(1-\bar{a}z)$, $S=\sum_{n=1}^{+\infty}(1-|a_n|)<+\infty$. Then we can find the sequence $\{r_n\}$*) such that

$$(1.1)\qquad \begin{aligned} &(1)\quad 1=r_1>r_2>r_3\cdots\cdots\to 0,\\ &(2)\quad \sum_{n=1}^{+\infty} 1/r_n^2\cdot(1-|a_n|)<+\infty.\end{aligned}$$

For the sake of convenience, we introduce some notations:

(1) $D(e^{i\varphi}, \vartheta)=\{z;\ |\arg(1-ze^{-i\varphi})|\leqq \vartheta<\pi/2,\ |z-e^{i\varphi}|\leqq\cos\vartheta\}$.

(2) $D(e^{i\varphi}, r_1, r_2)=(|z-r_1e^{i\varphi}|\leqq 1-r_1)\cap(|z-r_2e^{i\varphi}|\geqq 1-r_2)$, $(0<r_1<r_2<1)$.

(3) $\mathscr{D}=\bigcap_n \{z;\ \rho(z, a_n)\geqq R_n\}$, where $\rho(a, b)$: The non-Euclidean hyperbolic distance between a and b, $R_n=\tanh^{-1} r_n$ $(n=1, 2, \cdots)$.

(4) $S(\varepsilon)=\sum_{|a_n-e^{i\varphi}|<\varepsilon} 1/r_n^2\cdot(1-|a_n|)$.

Then we can state our theorems as follows:

Theorem 1.

$$(1.2)\qquad \lim_{\substack{|z|\to 1\\ z\in\mathscr{D}}}(1-|z|)\log|1/B(z)|=0.$$

As its immediate consequences, we get following:

Corollary 1.

(1.3) $\lim|z-e^{i\varphi}|\cdot\log|1/B(z)|=0$ *uniformly as* $z\to e^{i\varphi}$ *inside* $D(e^{i\varphi}, \vartheta)\cap\mathscr{D}$.

Corollary 2. *If there exists no* $\{a_n\}$ *in the sector* S: $\alpha\leqq\arg(1-ze^{-i\varphi})\leqq\beta(-\pi/2<\alpha<\beta<\pi/2)$, *then* $\lim|z-e^{i\varphi}|\cdot\log|1/B(z)|=0$ *uniformly as* $z\to e^{i\varphi}$ *inside the subsector of* S.

As an interesting application of Corollary 2, we can establish

Theorem 2. *If the sequence* $\{a_n\}$ *lies on the chord* $L:\arg(1-ze^{-i\varphi})=\vartheta$ $(|\vartheta|<\pi/2)$, *then* L *is Julia- line for* $f(z)=B(z)\cdot\exp\{\alpha\cdot(e^{i\varphi}+z)/(e^{i\varphi}-z)\}$ $(\alpha>0)$.

Under additional conditions, we can prove more precise theorems than Theorem 1:

Theorem 3. *If* $\overline{\lim}_{\varepsilon\to+0} S(\varepsilon)/\varepsilon^2<+\infty$, *then* $\underline{\lim}|B(z)|>0$ *as* $z\to e^{i\varphi}$ *inside* $\mathscr{D}\cap(D(e^{i\varphi}, \vartheta)\cup D(e^{i\varphi}, r_1, r_2))$.

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*) Vide lemma 1.