180. *Note on Permutability of Congruences on Algebraic Systems*

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In his paper [3, Theorem 4], A. I. Mal'cev gave a necessary and sufficient condition that all congruences should be permutable for every algebraic system of a primitive class: There must exist a derived composition $f(\xi, \eta, \zeta)$ (a function defined by iteration of the compositions) such that $f(\xi, \xi, \zeta)=\zeta$ and $f(\xi, \zeta, \zeta)=\xi$ are identities in each of this class. This condition is clearly equivalent to the following: Let $\mathfrak{F}$ be the algebraic system freely generated by $\{x, y, z\}$ in this class. If φ_1 and φ_2 are congruences on $\mathfrak{F}$ generated by the relations $x\equiv y$ and $y\equiv z$ respectively, then there exists a derived composition $f(\xi, \eta, \zeta)$ such that $f(x, y, z)$ and z are congruent modulo φ_1 and that $f(x, y, z)$ and x are congruent modulo φ_2. In his book [1, pp. 22–23], R. H. Bruck has stated that the Mal'cev's result does not apply to multiplicative quasigroups, and that the free quasigroup of rank 4 (and hence any free quasigroup of higher rank) has non-permutable multiplicative congruences, but the facts for free quasigroups of rank 1, 2, or 3 seem to be unknown, similarly for free loops of arbitrary positive rank.

In this note, we shall study generalizations of the above Mal'cev's result and the others. Theorem 1 is a generalization of the Mal'cev's result, which can apply to multiplicative quasigroups. By this theorem, we can easily obtain that the free quasigroup of rank 3 has non-permutable multiplicative congruences. Theorems 2 and 3 are similar generalizations of the analogous results [2, Theorems 1 and 2] for weak permutability and local permutability of congruences. These theorems can apply to multiplicative loops, and it can be easily seen that all multiplicative congruences on any loop are locally permutable.

Let $\mathfrak{A}$ be an algebraic system with respect to a system V of compositions, and let W be a subsystem of V. An equivalence relation θ on $\mathfrak{A}$ is called a W-congruence if and only if

$$w(a_1, a_2, \cdots, a_{N(w)}) \overset{\theta}{\sim} w(b_1, b_2, \cdots, b_{N(w)})^{1)}$$

holds for every composition w in W, and for all elements $a_1, a_2, \cdots,$

1) $x \overset{\theta}{\sim} y$ denotes that x and y are congruent modulo θ.