167. Monotone Sequence of 0-dimensional Subsets of Metric Spaces

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Let ω_1 be the first uncountable ordinal and $\omega(c)$ the first ordinal whose power is c. This paper proves the following two theorems.

Theorem 1. *Let X be a metric space which is the countable sum of 0-dimensional subsets. Then there exists a sequence $\{J_i : i < \omega\}$ of subsets of X such that* i) $J_i \subset J_{i+1}$ *for every* i, ii) dim $J_i \leq 0$ *for every* i, *and* iii) $\cup J_i = X$.

Theorem 2. *Let X be a non-empty metric space. Then there exists a transfinite sequence $\{J_\alpha : \alpha < \omega_1\}$ of subsets of X such that* i) $J_\alpha \subset J_\beta$ *whenever* $\alpha < \beta$, ii) dim $J_\alpha \leq 0$ *for every* α, *and* iii) $\cup J_\alpha = X$.

In both cases we use the following notations, where ρ is the preassigned metric on X. Take two sequences $\mathfrak{U}_{ij} = \{U_\lambda : \lambda \in A_{ij}\}$ and $\mathfrak{F}_{ij} = \{F_\lambda : \lambda \in A_{ij}\}$, where $i, j = 1, 2, \cdots$, which satisfy the following conditions (cf. Bing [1]):

(1) $\mathfrak{U}_{ij}$ is a discrete collection of open sets of X.

(2) $\mathfrak{F}_{ij}$ is a collection of non-empty closed sets of X.

(3) $F_\lambda \subset U_\lambda$ for every $\lambda \in A$, where $A = \underset{i\ j}{\cup} A_{ij}$.

(4) $\mathfrak{F}_i = \{F_\lambda : \lambda \in A_i\}$ covers X for every i, where $A_i = \underset{j}{\cup} A_{ij}$.

(5) $\mathfrak{U}_i = \{U_\lambda : \lambda \in A_i\}$ is locally finite.

(6) $\rho(\mathfrak{U}_i) < 1/i$.

Set $U_{ij} = \cup\{U_\lambda : \lambda \in A_{ij}\}$ and $F_{ij} = \cup\{F_\lambda : \lambda \in A_{ij}\}$.

Proof of Theorem 1. Let I' be the set of all rational numbers r with $0 < r < 1$. By Nagata [4, Lemma 4.1] there exists a collection $\{U_{ijr} : i, j = 1, 2, \cdots, r \in I'\}$ of open sets of X which satisfies the following conditions:

(7) $F_{ij} \subset U_{ijr} \subset \bar{U}_{ijr} \subset U_{ijs} \subset \bar{U}_{ijs} \subset U_{ij}$ for $r < s$.

(8) $\{B(U_{ijr}) = \bar{U}_{ijr} - U_{ijr} : i, j = 1, 2, \cdots, r \in I'\}$ is pointfinite.

Let $I' = \{r_1, r_2, \cdots\}$ and $J_i = X - \cup\{B(U_{jkr}) : j, k = 1, 2, \cdots, r \in I' - \{r_1, \cdots, r_i\}\}$. Then by Morita [3, Lemma 3.3] dim $J_i \leq 0$. It is evident that $J_1 \subset J_2 \subset \cdots$. To see $\cup J_i = X$ let x be an arbitrary point of X. By (8) there exists i such that $x \notin B(U_{ijr})$ for any i, j and any $r \in I' - \{r_1, \cdots, r_i\}$. Hence $x \in J_i$ and the proof is completed.

Proof of Theorem 2. Let I be the unit interval $[0, 1]$ and $\{I_\alpha : \alpha < \omega(c)\}$ the family of all residue classes of I *modulo* the rational numbers. Set

$$L_\alpha = \cup\{I_\beta : \beta \leq \alpha \text{ or } \omega_1 \leq \beta < \omega(c)\}, \quad \alpha < \omega_1.$$