59. *Fundamental Equations of Branching Markov Processes*

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We have given in the previous paper [2] a definition of branching Markov processes and discussed some fundamental properties of them. Here we shall treat several fundamental equations which describe and characterize these processes.

1. Fundamental quantities of branching Markov processes. In this paper we shall use constantly the notation[1] and the terminolog adopted in [2].

Definition 1.1. Let X_t be a branching Markov process (abbreviated as B.M.P.) on $\boldsymbol{S}$. We denote the killed process on S^n of X_t at the first branching time τ by $X_t^{(0)}$ and call it the *non branching part* on S^n of B.M.P. X_t. The non branching part on S^1 is called simply the non branching part of X_t, and its semi-group on $\boldsymbol{B}(S)$ is defined by

(1.1) $$T_t^0 f(x) = E_x[f(X_t); t<\tau], \qquad f\in \boldsymbol{B}(S), x\in S.$$

Further we denote

(1.2) $$K(x, dt, dy) = P_x[\tau\in dt, X_{\tau-}\in dy], \qquad x\in S, dy\subset S.$$[2]

Definition 1.2. Assume that there exists a system $\{q_n(x); n=0,2,3,\cdots,+\infty\}$ of non-negatives Borel measurable functions on S and a system $\{\pi_n(x, d\boldsymbol{y}); n=0,2,\cdots,+\infty\}$ of non-negatives kernels[3] on $S\times \boldsymbol{S}$ such that

(1.3) $$P_x[X_\tau \in d\boldsymbol{y} \mid X_{\tau-}] = \pi(X_{\tau-}, d\boldsymbol{y}),$$

almost surely (P_x) on $\{\tau<\infty\}$, $x\in S, d\boldsymbol{y}\subset \boldsymbol{S}$, where we put

(1.4) $$\pi(x, d\boldsymbol{y}) = \sum_{n=0}^{\infty}{}^{*}\, q_n(x)\pi_n(X, d\boldsymbol{y}\cap S^n),$$

and $\sum_{n=0}^{\infty}{}^{*}$ denotes the sum over $n=0, 2, \cdots, +\infty$ and $S^\infty=\{\Delta\}$. Then we shall call $\{q_n, \pi_n, n=0, 2, \cdots, +\infty\}$ the *branching system* of B.M.P. X_t. It is clear that if a kernel $\pi(x, d\boldsymbol{y})$ on $S\times\boldsymbol{S}$ satisfing (1.3) is given, then the system

(1.5) $$q_n(x)=\pi(x, S^n),\ \pi_n(x, d\boldsymbol{y})=\pi(x, d\boldsymbol{y})/q_n(x), \qquad n=0, 2, \cdots, +\infty,$$

is the branching system of B.M.P. X_t.

The above defined $\{T_t^0, K, q_n, \pi_n\}$ are fundamental quantities of B.M.P. which completely determine the B.M.P. X_t. In this paper

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1) In [2], branching Markov processes are denoted by x_t, but in the following we write it as X_t.

2) We write as $X_{\sigma-}=\lim_{t\uparrow\sigma} X_t$, for any random time σ.

3) $\pi(x, d\boldsymbol{y})$ is said to be a non-negative kernel on $S\times\boldsymbol{S}$, if for any Borel set $B\subset\boldsymbol{S}$, $\pi(., B)$ is a Borel measurable function on S and for any $x\in S$, $\pi(x, .)$ is a non-negative measure on $\boldsymbol{S}$ with total mass less than 1.