58. On Conditions for the Orthomodularity

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1. Introduction. The lattice of projections of a von Neumann algebra is an orthocomplemented lattice (a lattice equipped with an orthocomplementation $a \to a^{\perp}$) with a weak modularity (M) introduced by Loomis [2]. Such a lattice is called an orthomodular lattice (see [3], Remark 4.1). The condition (M) for the orthomodularity is equivalent to that "if $a \leqq b$ then $a, a^{\perp}, b$ satisfy some distributive relation". Piron [5] has shown that the logic of quantum mechanics forms an orthomodular lattice by the reason that "if $a \leqq b$ then the sublattice generated by $a, a^{\perp}, b, b^{\perp}$ is distributive". This condition is also equivalent to (M).

On the other hand, Nakamura [4] has defined the permutability of a, b by some distributive relation and proved that the condition (M) is equivalent to that this permutability is symmetric. Moreover, Foulis [1] has given some other conditions like this.

The purpose of this paper is to find all the conditions of these types.

2. D-relations. Let L be an orthocomplemented lattice where the orthocomplementation is denoted by $a \to a^{\perp}$. For $a, b, c \in L$, we write $(a, b, c)D$ in case $(a \cup b) \cap c = (a \cap c) \cup (b \cap c)$, and write $(a, b, c)D^*$ in case $(a \cap b) \cup c = (a \cup c) \cap (b \cup c)$.

Definition. Two elements $a, b \in L$ are said to be *commutative* if the sublattice generated by $a, a^{\perp}, b, b^{\perp}$ is distributive. We denote aDb if every distributive relation for $a, a^{\perp}, b, b^{\perp}$ holds. (Obviously, if a and b are commutative then aDb.) Since $(a, b, c)D \Longleftrightarrow (b, a, c)D$ and $(a, b, c)D^* \Longleftrightarrow (a^{\perp}, b^{\perp}, c^{\perp})D$ for every $a, b, c \in L$, aDb is equivalent to that the following twelve D-relations hold.

$D_1 : (a, a^{\perp}, b)D$	$D_{13} : (b^{\perp}, a^{\perp}, a)D$	$D_{14} : (b^{\perp}, a, a^{\perp})D$
$D_2 : (a, a^{\perp}, b^{\perp})D$	$D_{23} : (b, a^{\perp}, a)D$	$D_{24} : (b, a, a^{\perp})D$
$D_3 : (b, b^{\perp}, a)D$	$D_{31} : (a^{\perp}, b^{\perp}, b)D$	$D_{32} : (a^{\perp}, b, b^{\perp})D$
$D_4 : (b, b^{\perp}, a^{\perp})D$	$D_{41} : (a, b^{\perp}, b)D$	$D_{42} : (a, b, b^{\perp})D$

Lemma 1. *D_i implies D_{ij}* ($i=1, 2$ and $j=3, 4$; $j=3, 4$ and $j=1, 2$).

Proof. D_1 means $b = (a \cap b) \cup (a^{\perp} \cap b)$. From this, we have $b \cup a^{\perp} = (a \cap b) \cup a^{\perp}$, $b \cup a = (a^{\perp} \cap b) \cup a$, and hence $b^{\perp} \cap a = (a^{\perp} \cup b^{\perp}) \cap a$, $b^{\perp} \cap a^{\perp} = (a \cup b^{\perp}) \cap a^{\perp}$ by the orthocomplementation. Therefore, D_{13} and D_{14} hold. The other implications can be proved similarly.