218. The Separable Axiomatization of the Intermediate Propositional Systems S_n of Gödel

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In [3] Gödel introduced a series of many-valued propositional systems S_n, which is widely known and is quite frequently made use of when propositional systems are treated. And in our paper [6] we introduced two kinds of axiomatization for these S_n. But the *separation theorem* mentioned below does not hold on those axiomatized systems.

Separation Theorem. *A provable formula in the system can be proved using only the axioms for implication and those for the logical symbols actually appearing in the formula.*

We introduce, in this paper, another axiomatization for S_n and prove the separation theorem on them.

§1. Preliminaries. Definition 1.1. *S_n is a many-valued propositional system, whose values are integers $1, 2, \cdots, n$ and ω (ω is regarded greater than any positive integers), and whose sole designated value is 1. Logical operations $\supset$, $\wedge$, $\vee$, and $\neg$ are defined in S_n as follows:*

$$v_1 \supset v_2 = \begin{cases} 1 & \text{if } v_1 \geqq v_2, \\ v_2 & \text{otherwise}, \end{cases}$$
$$v_1 \wedge v_2 = \max(v_1, v_2),$$
$$v_1 \vee v_2 = \min(v_1, v_2),$$
$$\neg v = v \supset \omega.$$

An extension of S_n is $\boldsymbol{LC}$ of Dummett [2], in which values are defined to be all the positive integers and ω.

By $S \vdash A$, we mean that a formula A is provable (or valid) in the axiomatic system (or model) S. By $S + A_1 + \cdots + A_k$, we mean an axiomatic system obtained by adding the axiom schemes $A_1, \cdots, A_k$ to an axiomatic system S. If S_1 and S_2 are two systems axiomatic or defined by a model, we mean by $S_1 \supset S_2$ that the set of all provable or valid formulas of S_2 is included in that of S_1. And $S_1 \supset\subset S_2$ means that $S_1 \supset S_2$ and $S_2 \supset S_1$. If f is an assignment function of a model, we mean by $f(A)$ the value calculated for the formula A by the assignment f.

Lemma 1.2. $S_1 \supset S_2 \supset \cdots \supset S_n \supset \cdots \supset \boldsymbol{LC} \supset \boldsymbol{LI}$, *where $\boldsymbol{LI}$ is the intuitionistic system and S_1 coincides with the usual classical*