# 7. *On Hausdorff's Theorem*

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In his paper [2], Professor T. Satō considers directed sequences of real numbers, and the Riemann-Stieltjes integral as its application.

In the case of the Riemann-Stieltjes integral, he generalizes Darboux's theorem on the Riemann integral and obtains the following two theorems:

**Theorem 1.** *Let* $\{\psi_n(x)\}$ *be a sequence of bounded functions in* $[a, b]$.

*If* $\psi_1(x) \geqq \psi_2(x) \geqq \cdots \geqq \psi_n(x) \geqq \cdots$, *and*

$$\lim_{n\to\infty} \psi_n(x) = 0,$$

*then*

$$\lim_{n\to\infty} \int_a^b \psi_n(x) d\sigma(x) = 0.$$

**Theorem 2.** *Let* $\{f_n(x)\}$ *be a sequence of uniformly bounded functions in* $[a, b]$.

*If a sequence of functions* $f_n(x)$ $(n=1, 2, \cdots)$ *converges to a function* $f(x)$, *then*

$$\overline{\lim_{n\to\infty}} \underline{\int_a^b} f_n(x) d\sigma(x) \leqq \overline{\int_a^b} f(x) d\sigma(x),$$

$$\underline{\lim_{n\to\infty}} \overline{\int_a^b} f_n(x) d\sigma(x) \geqq \underline{\int_a^b} f(x) d\sigma(x).$$

We shall generalize the latter using his method.

In this note, we shall prove the following theorem which is a generalization of the theorem 2.

**Theorem.** *Let* $\{f_n(x)\}$ *be a sequence of uniformly bounded functions in* $[a, b]$.

*Let* $\underline{f}(x) = \underline{\lim}_{n\to\infty} f_n(x)$, $\bar{f}(x) = \overline{\lim}_{n\to\infty} f_n(x)$, *then we have*

$$\overline{\lim_{n\to\infty}} \underline{\int_a^b} f_n(x) d\sigma(x) \leqq \overline{\int_a^b} \bar{f}(x) d\sigma(x),$$

$$\underline{\lim_{n\to\infty}} \overline{\int_a^b} f_n(x) d\sigma(x) \geqq \underline{\int_a^b} \underline{f}(x) d\sigma(x).$$

To prove the theorem above, we shall first explain some notations.

Let $\sigma(x)$ be a continuous and strictly increasing function in $[a, b]$. We subdivide the interval $[a, b]$ by means of the points $x_0, x_1, \cdots, x_{n-1}, x_n$, so that