20. A Proof for the Imbedding Theorems for Sobolev Spaces

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(Comm. by Zyoiti SUETUNA, M.J.A., Feb. 13, 1967)

The purpose of the present paper is to show that we can give another proof for the imbedding theorems for Sobolev spaces, without making use of the customary estimate of multi-dimensional potentials ([1]-[3], [6]-[9]).

Theorem. *Let Ω be a domain in the n-dimensional Euclidean space $\boldsymbol{R}^n$. Assume that there exists a constant R, a cube Q and an open covering $\{\Omega_j\}$ of Ω such that the diameter of Ω_j is not greater than R, and Ω_j is star-shaped with respect to a cube Q_j congruent to Q. In cases $m>0$, assume that for each point x in Ω the number of Ω_j containing x is not greater than a constant N. Then there holds the imbedding:*

$$W^{l,p}(\Omega)\longrightarrow W^{k,q,n-m}(\Omega)$$

if $l-\frac{n}{p}+\frac{m}{q}-k\geqq 0$, $1<p\leqq q<\infty$ *and if either one of the conditions*

(i) $p<q, m>0$, (ii) *k is not an integer*, $m=0$, *and* (iii) $m=n$, *is satisfied.*

For functions $f(x)$ on Ω, we define

$$\|f\|_{L^{p,n-m}(\Omega)}=\sup_{x^{(m)}}\|f(x_1,\cdots,x_m,x^{(m)})\|_{L^p(\Omega_{x^{(m)}})},$$

where $x^{(m)}=(x_{m+1},\cdots,x_n)$ and $\Omega_{x^{(m)}}$ is the set of points $(x_1,\cdots,x_m)$ such that $(x_1,\cdots,x_m,x^{(m)})\in\Omega$, and for $f\in C^\infty(\Omega)$

$$\|f\|_{W^{l,p,n-m}(\Omega)}=\|f\|_{L^{p,n-m}(\Omega)}+\sum_{|\alpha|=l}\|f^{(\alpha)}\|_{L^{p,n-m}(\Omega)}\quad (l \text{ is an integer.})^{*)}$$

or

$$\|f\|_{W^{l,p,n-m}(\Omega)}=\|f\|_{W^{[l],p,n-m}(\Omega)}+\sum_{|\alpha|=[l]}\left\|\frac{f^{(\alpha)}(x)-f^{(\alpha)}(y)}{|x-y|^{(m/p)+\sigma}}\right\|_{L^{p,2n-2m}(\Omega\times\Omega)}$$

$$(l=[l]+\sigma,\ 0<\sigma<1)$$

Here the *spaces* $W^{l,p,n-m}(\Omega)$ *are defined* as the completions of subsets of $C^\infty(\Omega)$ consisting of functions f with $\|f\|_{W^{l,p,n-m}}<\infty$. $W^{l,p}=W^{l,p,0}$, $C^l=W^{l,p,n}$.

In the following we assume that Ω is bounded and is starshaped with respect to a cube. It is easy to extend the results for general domains.

Let Ω be a bounded star-shaped domain with respect to a cube

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*) $f^{(\alpha)}(x)$ denotes the α-th derivative of $f(x)$.