41. *Integration with Respect to the Generalized Measure. I*

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1. Introduction. In this paper, we are going to deal with the integration theory with respect to the *topological-additive-group-valued measure* [1].

Let M be a set and $\mathcal{S}$ a ring of subsets of M ($\mathcal{S}$ is a ring in the algebraic sense,[1] of which each element is an idempotent). Let μ be a *measure* [1] defined on $\mathcal{S}$ taking values in a topological additive group G.

Let K be a topological additive group and let $\mathcal{F}$ be the additive group of all K-valued functions defined on M (the sum of two functions in $\mathcal{F}$ is defined in the usual way).

For $X \in \mathcal{S}$ and $f \in \mathcal{F}$, let us denote by Xf the function in $\mathcal{F}$ such that

$$(Xf)(x)=\begin{cases} f(x) & \text{if} \quad x \in X, \\ 0 & \text{if} \quad x \in M-X. \end{cases}$$

Then each $X \in \mathcal{S}$ operates as a homomorphism on the group $\mathcal{F}$. We further assume that $\mathcal{F}$ is a topological group with some topology such that each $X \in \mathcal{S}$ operates as a continuous map on $\mathcal{F}$.

Let J be a topological additive group and suppose that a map of $G \times K$ into J, denoting by $g \cdot k$ the image of (g, k), $g \in G$, $k \in K$, is defined, satisfying the conditions:

1) $(g+g') \cdot k = g \cdot k + g' \cdot k$,
2) $g \cdot (k+k') = g \cdot k + g \cdot k'$,

for each $g, g' \in G$ and $k, k' \in K$.

As an illustration, suppose that M is the real line and $G=K=J$ is the topological ring of all real numbers. Let $\mathcal{S}$ be the *pseudo-σ-ring* [1] of measure[2]-finite Lebesgue measurable sets and μ the Lebesgue measure on $\mathcal{S}$ (strictly, its restriction on $\mathcal{S}$). Now we can consider $\mathcal{F}$ as a topological additive group introducing the topology in such a way that a sequence of functions in $\mathcal{F}$ converges in the space $\mathcal{F}$ if and only if the sequence uniformely converges as a functional sequence. Then, each $X \in \mathcal{S}$ operates as a continuous homomorphism of $\mathcal{F}$ into itself.

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1) $X+Y=(X-Y)\cup(Y-X)$, $XY=X\cap Y$ for each $X, Y \in \mathcal{S}$.

2) Lebesgue measure.