128. On the Convergence Criterion of M. Izumi and S. Izumi

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1. Introduction. Let $f(x)$ be a periodic function with period 2π and L-integrable over $[-\pi, \pi]$, and let

$$\phi(u)=\phi_x(u)=f(x+u)+f(x-u)-2f(x).$$

The following theorem on the convergence of Fourier series has been established by M. Izumi and S. Izumi [1]:

Theorem A. If

$$\int_0^t \phi(u)du=o(t) \quad (t\to 0), \tag{1}$$

and for some $\delta>0$, there is an $\alpha(0<\alpha<1)$ such that

$$\int_t^\delta |d(u^{-\alpha}\phi(u))|=o(t^{-\alpha}), \tag{2}$$

then the Fourier series of $f(x)$ converges to $f(x)$ at the point x.

This theorem is an extension of the following theorem of Tomic [2]:

Theorem B. If at the point x, $\phi(u)\to 0$ as $u\to 0$ and for $u\to 0$, $\phi(u)$ is slowly varying, then the Fourier series of $f(x)$ converges to $f(x)$ at the point x.

The aim of this paper is to discuss the relations between Izumi-Izumi's test and the following two tests:

Theorem C (Young). If (1) holds and

$$\int_0^t |d(u\phi(u))|=o(t), \tag{3}$$

then the Fourier series converges to $f(x)$.

Theorem D (Lebesgue). (1) and

$$\lim_{k\to\infty}\limsup_{t\to 0}\int_{kt}^{\pi}\frac{|\phi(u)-\phi(u+t)|}{u}du=0 \tag{4}$$

imply the convergence of the Fourier series of $f(x)$ at the point x.

We shall prove that Izumi-Izumi's test includes Young's but is included in Lebesgue's test.

2. The relation between M. Izumi - S. Izumi's and Young's test. We first prove that Izumi-Izumi's test includes Young's. It is enough to prove the following

Theorem 1. (3) *implies* (2).

Proof. Suppose that (3) holds. Then