167. On Closed Mappings and M-Spaces. II

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(Comm. by Kinjirô KUNUGI, M.J.A., Oct. 12, 1967)

1. Introduction. The main purpose of this paper is to give the affirmative answer to an open problem raised by A. Arhangel'skii in his recent communication to K. Morita whether the image Y under a perfect mapping f of a paracompact normal M-space X is an M-space or not.[1] A closed continuous mapping f of a topological space X onto a topological space Y is said to be perfect if the inverse images under f of points y of Y are compact subspaces of X. We shall prove the following main theorem.

Theorem 1.1. *Let f be a closed continuous mapping of an M-space X onto a normal space Y, where X is T_1. If $f^{-1}(y)$ is countably compact for any point y of Y, then Y is also an M-space.*

As a direct consequence of Theorem 1.1 we obtain the following

Cororally 1.2. *Let f be a closed continuous mapping of a normal M-space X onto a topological space Y, where X is T_1. If $f^{-1}(y)$ is countably compact for any point y of Y, then Y is also a normal M-space.*

Some applications and a generalization of our main theorem will be mentioned in §4.

2. Lemmas. Lemma 2.1. *Let T be a metric space. If $\{\mathfrak{F}_n\}$ is a sequeuce of locally finite closed coverings of T such that $\{\mathfrak{F}_n\}$ satisfies the condition $(*)$ and that $\mathfrak{F}_{n+1}$ is a refinement of $\mathfrak{F}_n$ for every n, then there exists a sequence $\{\mathfrak{U}_{nm} \mid n=1, 2, \cdots; m=1, 2, \cdots\}$ of locally finite open coverings of T such that*

(1) *$\{\mathfrak{U}_{nm}\}$ satisfies the condition $(*)$,*

(2) *$F_{n\lambda} \subset U_{nm\lambda}$ for $\lambda \in \Lambda_n$; $n=1, 2, \cdots$, $m=1, 2, \cdots$,*

where $\mathfrak{F}_n=\{F_{n\lambda} \mid \lambda \in \Lambda_n\}$ and $\mathfrak{U}_{nm}=\{U_{nm\lambda} \mid \lambda \in \Lambda_n\}$.

Proof. For any $F_{n\lambda}$ of $\mathfrak{F}_n$, let us put

$$V_{nm\lambda}=\{x \mid d(x, F_{n\lambda})<1/m\},$$

where d is a metric function in T and m is an arbitrary positive integer. Clearly $F_{n\lambda} \subset V_{nm\lambda}$. Let us put further

$$\mathfrak{V}_{nm}=\{V_{nm\lambda} \mid \lambda \in \Lambda_n\}.$$

Then we can prove that $\{\mathfrak{V}_{nm}\}$ satisfies the condition $(*)$. Indeed, let $\mathfrak{K}^k=\{K_i \mid i=1, 2, \cdots\}$ be a family of subsets of T which has the finite intersection property and contains as a member a subset of

1) Prof. K. Morita has kindly informed me of this open problem.