160. *A Characterization of Lukasiewiczian Algebra. I*

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In his papers [1], [2], using an algebraic technique, Prof. K. Iséki gave a characterisation of Boolean algebra. In this paper, I shall give a characterization of three-valued Lukasiewicz algebras, which were introduced by Prof. Gr. C. Moisil [3] as models for J. Lukasiewicz three-valued propositional calculus [4].

A L-algebra is a system $\langle X, 0, *, \sim\rangle$ where 0 is an element of a set X, $*$ is a binary operation and $\sim$ is a unary operation on X such that the axioms given below hold. We write $x\leq y$ for $x*y=0$, and $x=y$ for $x\leq y$ and $y\leq x$.

$L1)$ $x*y\leq x$,

$L2)$ $(x*y)*(x*z)\leq z*y$,

$L3)$ $x*(x*(z*(z*y)))\leq z*(z*(y*(y*x)))$,

$L4)$ $(x*z)*((x*z)*(y*z))\leq (y*z)*(y*x)$,

$L5)$ $x\leq x*(\sim x*x)$,

$L6)$ $x*(x*\sim x)\leq \sim(y*(y*\sim y)))$,

$L7)$ $x*\sim y\leq y*\sim x$,

$L8)$ $\sim x*y\leq \sim y*x$,

$L9)$ $0\leq x$.

Further we shall prove some proposition from the axioms $L1$—$L9$.

If we substitute $y*z$ for z in $L2$, then by $L1$, $L9$ we have

$$x*y\leq x*(y*z). \tag{1}$$

In (1) if we put $y=x$, $z=\sim x*x$ and use $L5$, $L9$, then we have

$$x*x=0. \tag{2}$$

By $L1$, $L9$ we have

$$0*x=0. \tag{3}$$

In $L3$ put $z=0$, then by (3), $L2$ we have

$$x=x*0. \tag{4}$$

By $L2$ we have the following lemmas.

Lemma 1. *$x\leq y$ implies $z*y\leq z*x$.*

Lemma 2. *$x\leq y$ and $y\leq z$ imply $x\leq z$.*

Let us put $z=y$ in $L3$, then by $L1$, (2), (4), Lemma 2, we have

$$x*(x*y)\leq y. \tag{5}$$

By $L2$ and Lemma 1 we have

$$u*(z*y)\leq u*((x*y)*(x*z)). \tag{6}$$

In (6) put $x=x*u$, $z=x*z$, $u=((x*u)*y)*(z*u)$ then