2. An Extension of Beurling's Theorem. I

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Let R be a Riemann surface with positive boundary and let $\{R_n\}$ $(n=0, 1, 2, \cdots)$ be its exhaustion with compact relative boundary ∂R_n such that $\partial R_n \cap \partial R_{n+1}=0$. Let $N(z, p)$ be a positive harmonic function in $R-R_0-p: p \in R-R_0$ such that $N(z, p)=0$ on ∂R_0, $N(z, p)$ has a logarithmic singularity at p and $N(z, p)$ has minimal Dirichlet integral over $R-R_0$, where Dirichlet integral is taken with respect to $N(z, p)$ $+\log|z-p|$ in a neighbourhood of p. We call such $N(z, p)$ an N-Green's function with pole at p. Consider now a sequence of points $\{p_i\}$ of $R-R_0$ having no points of accumulation in $R-R_0+\partial R_0$. Since the functions $N(z, p_i)$ $(i=1, 2, \cdots)$ forms, from some i on, a bounded sequence of harmonic functions—thus a normal family. A sequence of these functions, therefore is convergent in every compact part of $R-R_0$ to a positive harmonic function. A sequence $\{p_i\}$ of $R-R_0$ having no point of accumulation in $R-R_0+\partial R_0$, for which the corresponding $\{N(z, p_i)\}$ have the property just mentioned, that is, $\{N(z, p_i)\}$ converges to a harmonic function—will be called fundamental. If two fundamental sequences determine the same limit function $N(z, p)$, we say that they are equivalent. Two fundamental sequences equivalent to a given one determine an ideal boundary point of R. The set of all the ideal boundary points of R will be denoted by B and the set $R-R_0+B$ by $\bar{R}-R_0$. The domain of definition of $N(z, p)$ may now be extended by writing $N(z, p)=\lim_i N(z, p_i)$ $(z \in R-R_0,\ p \in \bar{R}-R_0)$, where $\{p_i\}$ is any fundamental sequence determining p. The function $N(z, p)$ is characteristic of the point p of their corresponding $N(z, p)$ as a function of z. The distance $\delta(p_1, p_2)$ of two points p_1 and p_2 in $\bar{R}-R_0$ is defined as

$$\delta(p_1, p_2)=\sup_{z \in R_1}\left|\frac{N(z, p_1)}{1+N(z, p_1)}-\frac{N(z, p_2)}{1+N(z, p_2)}\right|.$$

The topology (N-Martin's topology) [1] is induced by this metric.

Let $U(z)$ be a positive superharmonic function in $R-R_0$ such that $D(\min(M, U(z)))<\infty$ for every M and $U(z)=0$ on ∂R_0. Let G be a domain [2] in $R-R_0$ and let ${}_GU^M(z)$ be a superharmonic function in $R-R_0$ such that ${}_GU^M(z)=\min(M, U(z))$ on $G+\partial R_0$ and ${}_GU^M(z)$ has minimal Dirichlet integral. Put ${}_GU(z)=\lim_{M=\infty} {}_GU^M(z)$. If for any domain G, ${}_GU(z) \leqq U(z)$, $U(z)$ is called a full-superharmonic function