17. *A Condition of Convergent Filters*

By Tomoko KONDA

(Comm. by Kinjirô KUNUGI, M. J. A., Feb, 12, 1968)

The object of this note is to give definitions of convergent filter which satisfy or don't satisfy a condition [1].

A family $\mathcal{F}$ of subsets of a set is said to be a *filter* if it possesses the following properties:

(1) The void set ϕ is not in $\mathcal{F}$,

(2) If $A\supseteq B$ and $B\in\mathcal{F}$, then $A\in\mathcal{F}$,

(3) If $A, B\in\mathcal{F}$, then $A\cap B\in\mathcal{F}$.

If $\mathcal{F}$ and $\mathcal{F}'$ are filters of subsets of a set, we say that $\mathcal{F}$ is finer than $\mathcal{F}'$ if $\mathcal{F}\supseteq\mathcal{F}'$. A filter is called an *ultrafilter* if it is not fined by any filter but itself. The collection of all subsets of a topological space X which contain a neighborhood of a point p forms a filter $\mathfrak{N}(p)$. In general, a filter $\mathcal{F}$ of subsets of a topological space X converges to a point $p\in X$, if every neighborhood of p belong to $\mathcal{F}$. Thus the filter $\mathcal{F}$ converges to p if and only if $\mathcal{F}$ is finer than $\mathfrak{N}(p)$ (see [2]).

Lemma. *Let $x\in E$, then the collection $\dot{x}$ of all subsets of X which contain a point x is an ultrafilter on E.*

Definition. If a filter $\mathcal{F}$ is not convergent to x, then there exists $V\subseteq E$, $x\in V$, and $V\notin\mathcal{F}$ such that for each $y\in V$, $\mathfrak{G}$ converging to y includes V. When, we shall call that a set or a topological space E satisfies the condition (c).

Theorem 1. *Let E be a topological space, we define a filter $\mathcal{F}$ converge to x as $\mathcal{F}\supseteq\mathfrak{N}(x)$. i.e. $\mathcal{F}\to x\Longleftrightarrow\mathcal{F}\supseteq\mathfrak{N}(x)$, then the topological space E satisfies the condition (c). Moreover the inverse is true.*

Proof. Now we suppose $\mathcal{F}\not\to x$ and $\mathcal{F}\not\supseteq\mathfrak{N}(x)$, then there exists an open set $V\in\mathfrak{N}(x)$ in E, $V\notin\mathcal{F}$, and $x\in V$. Let $\mathfrak{G}\supseteq\mathfrak{N}(y)$ for each $y\in V$, of course, V is a neighborhood of y, therefore V is contained in $\mathfrak{G}$.

Conversely, we shall prove that if E satisfies the condition (c) then $\mathcal{F}\supseteq\mathfrak{N}(x)$ if and only if $\mathcal{F}\to x$. Let $\mathcal{F}\not\to x$, then there exists a set $V\subseteq E, x\in V$, $V\notin\mathcal{F}$ such that $V\in\mathfrak{N}(x)$. Hence $\mathcal{F}\not\supseteq\mathfrak{N}(x)$. On the other hand, let $\mathcal{F}\to x$, V be a open set in $\mathfrak{N}(x)$ then for each $y\in V$ and $\mathfrak{G}\to y$ we have $V\in\mathfrak{G}$. In particular, $x\in V, \mathcal{F}\to x$ so that $V\in\mathcal{F}$. Thus for each $V'\in\mathfrak{N}(x)$, there exists an open set V such that $V'\supseteq V$, $V\in\mathfrak{N}(x)$. By the above, the open set $V\in\mathcal{F}$ and $V'\in\mathcal{F}$ because $\mathcal{F}$ is a filter.

Theorem 2. *Let X, Y be topological spaces, and Φ a filter of*