16. On Criterion for the Nuclearity of Space $S\{M_p\}$

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In his paper [2], T. Yamanaka introduced a new type of function spaces $S\{M_p\}$ which includes $K\{M_p\}$ as well as all S-type spaces. In this note, we shall consider a criterion for the nuclearity of the space $S\{M_p\}$. The fundamental idea of its proof is essentially due to [1], [3]. For nuclear spaces and its related notion, see [1].

Let $M_p(x, q)(p=1, 2, \cdots)$ be functions defined for all $x \in R_n$ (n-dimensional Euclidean space) and all systems of n non-negative integers $q=(q_1, q_2, \cdots, q_n)$ which satisfy the following three conditions.

(1) $0 \leq M_1(x, q) \leq M_2(x,q) \leq \cdots \leq M_p(x, q) \leq \cdots$

(2) For every p there exists a positive number N_p which may be infinite, such that $\lim_{p\to\infty} N_p=\infty$ and $\inf_x M_p(x, q)>0$ for $|q|<N_p$ and $M_p(x, q)=0$ for $|q| \geq N_p$.

(3) For any fixed pair (x, q) there are only two possible cases; $M_p(x, q)=\infty$ for all p or $M_p(x, q)<\infty$ for all p.

Given such a system of functions $M_p(x, q)$, we denote by $S\{M_p\}$ the set of all infinitely differentiable functions $\varphi(x)$ for which the countable norms are finite, i.e.

$$\|\varphi\|_p=\sup_{x,q} M_p(x, q)|D^q\varphi(x)|<\infty.$$

Proposition 1. *The space $S\{M_p\}$ is complete.*

Proof of this proposition is found in ([2] or [3]).

We will say that a space $S\{M_p\}$ satisfies condition (N_1), if the following conditions hold.

(1) For any p there is $p' \geqq p$ such that the ratio

$$m_{pp'}(x)=\sup_q \frac{M_p(x, q)}{M_{p'}(x, q)} \qquad \left(\frac{0}{0}=\frac{\infty}{\infty}=0\right).$$

goes to zero as $|x|\to\infty$ and $m_{pp'}(x)$ is a summable function of x.

(2) If there exists q such that $M_p(x,q)\neq 0, \neq\infty$ for every $x \in R_n$, then we can obtain the following inequality:

$$M_p(x, q) \leq K_{pp'} M_{p'}(y, q+\alpha) \quad \text{for} \quad |y-x| \leq 1 \text{ and } |\alpha| \leq n,$$

where $K_{pp'}$ is a suitable constant number and n is a arbitrary positive integer. The following Lemma is due to [3]:

Lemma. *Let $\varphi(x)$ be a n-ordered continuous differentiable function on $B(x; r)$,*[1] *then we can obtain the following inequality* $|\varphi(x)| \leq A_r \sum_{|\beta|\leq n} \int_{|y-x|\leq r} |D^\beta\varphi(y)|\,dy$, *where A_r is a suitable constant*

1) $B(x; r)$ denotes the closed ball with center x and radius r.