9. *Ackermann's Model and Recursive Predicates*

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Let N be the set of all non-negative integers. Define a binary predicate $\in$ on N by

$$a \in b. \equiv . [b/2^a] \text{ is odd,}$$

where $[x]$ means the greatest integer contained in x. (For the recursive definition of $[x/y]$, see Kleene [1], p. 223). Then the structure $\langle N, \in \rangle$, which is called Ackermann's model, satisfies all the axioms of ZF except the axiom of infinity.

A predicate $P(a_1, \cdots, a_n)$ on N is called bounded, if there exists a restricted formula $A(x_1, \cdots, x_n)$ in the sence of [2] such that $P(a_1, \cdots, a_n)$ holds if and only if $A(a_1, \cdots, a_n)$ is true in $\langle N, \in \rangle$. Then our main theorem can be stated as follows:

Theorem. *A predicate $R(a_1, \cdots, a_n)$ is general recursive if and only if there exists bounded predicates $P(a, a_1, \cdots, a_n)$ and $Q(a, a_1, \cdots, a_n)$ such that*

(1) $\quad R(a_1, \cdots, a_n) \equiv \exists x P(x, a_1, \cdots, a_n) \equiv \forall x Q(x, a_1, \cdots, a_n)$

for all $a_1, \cdots, a_n \in N$.

Proof. First suppose that there exist P and Q satisfying (1). Since $\in$ is primitive recursive, we can easily show that every bounded predicate is primitive recursive. Hence, by the theorem VI(b) of [1], R is general recursive. Before proving the converse, we prove several lemmata. We temporarily call a predicate R for which there can be found bounded predicates P and Q satisfying (1) as a Δ-predicate.

Lemma 1. *$a<b$ is a Δ-predicate.*

Proof. Let $A(p, z). \equiv .$ $\mathrm{Comp}(z) \wedge p \subseteq z \times z \wedge \forall x \forall y (\langle xz \rangle \in p$ $\equiv x \in z \wedge y \in z \wedge \exists u (u \in y \wedge u \notin x \wedge \forall v (\langle uv \rangle \in p \supset (v \in x \equiv v \in y))))$, where $z \times z$ means direct product. Then $A(p, z)$ has the following properties:

1° $A(p, z)$ is bounded.

2° If $A(p, z)$, then we have

$$\forall i \forall j (\langle ij \rangle \in p \equiv i \in z \wedge j \in z \wedge i < j).$$

3° $\forall a \forall b \exists p \exists z (a \in z \wedge b \in z \wedge A(p, z))$.

1° and 3° are easily proved. 2° is proved by the induction on $\max(i, j)$. Therefore

$$a<b \equiv \overset{\forall}{\exists} p \overset{\forall}{\exists} z (a \in z \wedge b \in z \wedge A(p, z) \overset{\supset}{\wedge} \langle ab \rangle \in p).$$

This clearly shows $a<b$ is a Δ-predicate.

Lemma 2. *$a'=b$ is a Δ-predicate.*