32. On Generalized Integrals. I

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(Comm. by Kinjirô KUNUGI, M. J. A., March 12, 1968)

1. **Introduction.** Prof. K. Kunugi introduced, in 1954, the notion of ranked spaces, in the Note [2], as an extension of the metric space, and introduced further, in 1956, the notion of a generalized integral, in the Note [3], based on his theory of ranked spaces and called it the (E. R.) integral. In fact, to give the definition of his generalized integral, he started with the set $\mathcal{E}$ of step functions defined on a finite interval $a \leq x \leq b$, that is, functions having a constant value α_i in each of a finite number of sub-intervals $a_{i-1} < x < a_i$ in a division of $a \leq x \leq b$: $a_0 = a < a_1 \cdots < a_n = b$, as to the endpoints of these sub-intervals, we can assign values of the functions there arbitrarily. He supposed the integral defined for these functions, as usual, by the sum $\sum_i \alpha_i(a_i - a_{i-1})$. He introduced on the set $\mathcal{E}$ the set of neighbourhoods defined in the following way: Given a non-negative integer ν, a closed subset F of the interval $a \leq x \leq b$ and a point f of $\mathcal{E}$, the neighbourhood $V(F, \nu; f)$ of f is the set of all step functions $g(x)$ such that: $g(x) - f(x)$ is expressed as a sum of two step functions $p(x)$ and $\gamma(x)$ satisfying the following conditions:

[1] $\gamma(x) = 0$ for all $x \in F$,

[2] $\int_a^b |p(x)| dx < 2^{-\nu}$,

[3] $\left|\int_a^b \gamma(x) dx\right| < 2^{-\nu}$.

Under this topology, $\mathcal{E}$ becomes a uniform space the depth of which is ω_0, and so the indicator of $\mathcal{E}$ should be ω_0. The set $\mathfrak{B}_\nu$ of neighbourhoods of rank $\nu (\nu = 0, 1, 2, \cdots)$ is formed by the neighbourhoods $V(F, \nu; f)$ with mes $([a, b] \setminus F) < 2^{-\nu}$.[1] In this ranked space $\mathcal{E}$, we see that if $u: \{V(F_n, \nu_n; f_n)\}$ is a fundamental sequence of neighbourhoods, the limit $f(x) = \lim_{n\to\infty} f_n(x)$ exists almost everywhere, and the integrals $\int_a^b f_n(x) dx$ converges to a finit limit. This suggests that it should be possible to take this limit as the value of the integral of $f(x)$. As a formula

$$\int_a^b f(x) dx = \lim_{n\to\infty} \int_a^b f_n(x) dx.$$

To justify this convention, he showed that it does not depend on the particular choice of the fundamental sequence of neighbourhoods,

1) For the sets E and F, $E \setminus F$ denotes the set of all those points of E which do not belong to E.