31. On the Representations of SL(3, C). III

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In this part of the works we shall discuss unitary representations of G, including the supplementary series and the degenerate series.

1. It is already seen [1] that there exists the following invariant bilinear form on $\mathcal{D}_\chi \times \mathcal{D}_{\chi'}$ where $\chi=(l_1, m_1; \lambda_2, \mu_2)$ and $\chi'=(l_1, m_1; -l_1-\lambda_2, -m_1-\mu_2)$:

$$\int \delta^{(l_1, m_1)}(z_1')\varphi(z_1'z)\psi(z)dz_1'dz.$$

This form is degenerate, that is, if $\varphi \in \mathcal{E}_\chi^1$ or $\psi \in \mathcal{E}_{\chi'}^1$ we have $B(\varphi, \psi)=0$; moreover we obtain the following form on $\mathcal{E}_\chi^1 \times \mathcal{E}_{\chi'}^1$,:

$$B_1(\varphi, \psi)=(-1)^{p+q}p!(l_1-p-1)!q!(m_1-q-1)!$$

$$\times \int a_{pq}(z_2, z_3)b_{rs}(z_2, z_3)dz_2dz_3 \qquad (l_1-p-r-1=0 \text{ and } m_1-q-s-1=0),$$

$$=0 \qquad (l_1-p-r-1\neq 0 \text{ or } m_1-q-s-1\neq 0)$$

for $\varphi(z)=z_1^{(p,q)}a_{pq}(z_2, z_3)$ and $\psi(z)=z_1^{(r,s)}b_{rs}(z_2, z_3)$.

We remark that this form is equivalent to the non-degenerate form on $\mathcal{D}_{\chi s_1}/\mathcal{F}_{\chi s_1}^1 \times \mathcal{D}_{\chi' s_1}/\mathcal{F}_{\chi' s_1}^1$:

$$\int z_1'^{(l_1-1, m_1-1)}\varphi(z'z)\psi(z)dz_1'dz.$$

In particular, if $l_1=1$ and $m_1=1$, the representation $\{T^\chi, \mathcal{E}_\chi^1\}$ is the so-called degenerate representation and bilinear form on $\mathcal{E}_\chi^1 \times \mathcal{E}_{\chi'}^1$ is clearly given by

$$\int a(z_2, z_3)b(z_2, z_3)dz_2dz_3.$$

2. Now we set $\langle \varphi, \psi \rangle = B(\varphi, \bar{\psi})$ for $\varphi, \psi \in \mathcal{D}_\chi$, where $\bar{\psi}$ is the complex conjugate of ψ and $\bar{\psi} \in \mathcal{D}_{\bar{\chi}}$, then $\langle \cdot, \cdot \rangle$ is an Hermitian form on $\mathcal{D}_\chi$. In case it exists and is positive definite, the representation $R(\chi)$ is unitary with respect to this scalar product.

(i) When $\chi\bar{\chi}(\delta)=1$, that is, $\lambda_1=(n_1+\sqrt{-1}\rho_1)/2$, $\mu_1=(-n_1+\sqrt{-1}\rho_1)/2$, $\lambda_2=(n_2+\sqrt{-1}\rho_2)/2$, $\mu_2=(-n_2+\sqrt{-1}\rho_2)/2$, where n_k are integers and ρ_k are real, then $\langle \varphi, \psi \rangle$ has the form $\int \varphi(z)\bar{\psi}(z)dz$ and is positive definite. Such representations are known as those of the principal series.

(ii) When $\chi\bar{\chi}^{s_1}(\delta)=1$, that is, $\lambda_1=\mu_1=\sigma$, $\lambda_2=-\sigma/2+(n-\sqrt{-1}\rho)/2$, $\mu_2=-\sigma/2+(-n-\sqrt{-1}\rho)/2$, where n is an integer, σ and ρ are