67. *Characterizations of Self-Injective Rings*

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In the theory of (non-commutative) rings, self-injective rings are one of the most attractive objects, and have been studied in the last two decades by many authors. It is well known that a ring R with identity element is right self-injective if and only if, for each right ideal I and for each map $f: I_R \to R_R$, there exists $a \in R$ such that $f(i) = ai$ for all $i \in I$ (See Baer [1, Theorem 1]). The theory of QF-rings provides us with many characterizations of self-injective rings with minimum condition. For example, the following conditions are equivalent for a (left or right) Artinian ring R:

(1) *R is right self-injective.*

(2) *$l(r(L))=L$, $r(l(I))=I$ for each left ideal L and right ideal I.*

(3) *If aR (resp. Ra), $a \in R$, is simple then $l(r(a))=Ra$ (resp. $r(l(a)) = aR$).*

For a discussion of the condition (3), see Kato [6, Lemma 2].

In this paper we shall give some characterizations of right self-injective rings in terms of duality.

1. Preliminaries. Throughout this paper each ring R will be a ring with identity element and each module over R will be unital.

If A is a right R-module, let $A^* = \mathrm{Hom}_R(A, R)$ be its dual and let $\delta_A: A \to A^{**}$ be the natural map. We call, as usual, A torsionless (resp. reflexive) if δ_A is a monomorphism (resp. an isomorphism). If X is a subset of A (resp. A^*), then we set

$$l(X)=\{b \in A^* \mid bX=0\} \text{ (resp. } r(X)=\{a \in A \mid Xa=0\}).$$

We shall have need of the following lemma for our characterizations of right self-injective rings.

Lemma 1. **(Rosenberg and Zelinsky [7, Theorem 1.1]).** *Let R be a right self-injective ring, A a right R-module, and B a finitely generated submodule of A^*. Then $l(r(B))=B$.*

Proof. Write $B=Rb_1+\cdots+Rb_n$, $b_i \in B$, and let $b \in l(r(B))$. Then $\bigcap_{i=1}^{n} r(b_i)=r(B) \subset r(b)$. Hence there exists a map $f: \overset{n}{\oplus} R_R \to R_R$ such that $(b_1a, \cdots, b_na) \to ba$, $a \in A$, by virtue of the injectivity of R_R. Then

$$ba = f(b_1a, \cdots, b_na) = f(b_1a, 0, \cdots, 0) + \cdots + f(0, \cdots, 0, b_na)$$
$$= r_1b_1a + \cdots + r_nb_na,$$