109. A Characterization of Best Tchebycheff Approximations in Function Spaces

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1. Introduction. The purpose of this paper is to prove a theorem which characterizes best Tchebycheff approximations in linear subspaces of the space of all continuous real- or complex-valued functions defined on a compact topological space. There is a recent theorem due to I. Singer [1, Theorem 5] which treats the identical problem. The theorem presented in this paper differs from his in that we characterize best approximations in terms of the evaluation functionals corresponding to those points (the critical points) at which the error function (i.e., approximant-approximator) attains its maximum deviation from zero. Those evaluation functionals are, incidentally, precisely the extreme points of the unit sphere of the conjugate space [3, Problem J, p. 134]. The present theorem is also a direct generalization of a known theorem which treats the case where the approximating functions form a finite-dimensional subspace (see, for example, [4]). We shall state this special case as a corollary to the main theorem.

Actually, the main theorem on the part of necessity was first proved for reflexive subspaces. It is Dr. E. W. Cheney who kindly showed in his letter to the author that the original proof could be improved with a slight modification to include all linear subspaces. The author would like to express his gratitude to Dr. Cheney for this and other valuable suggestions.

Let X be a given compact topological space; $C(X)$, the space of all real- or complex-valued continuous functions defined on X with the Tchebycheff norm:

$$||f|| = \max\{|f(x)| : x \in X\}.$$

Let M be a proper subspace of $C(X)$; x^*, the evaluation functional corresponding to an element x in X, that is,

$$x^*(f) = f(x) \text{ for all } f \text{ in } C(X).$$

2. Main theorem. Let f be an element of $C(X)\backslash M$ and p an element of M. Let r denote the error f-p. Denote by K the set of

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