102. *Integration with Respect to the Generalized Measure. III*

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1. Introduction. Suppose M, $\mathcal{S}$, G, K, J, $\mathcal{F}$, $\mathcal{G}$, μ, and $\mathcal{J}$ are defined as are in the example in the introduction in [1]. Then $(\mathcal{S}, \mathcal{G}, J)$ is an abstract integral structure [1] and $\mathcal{J}$ is an abstract integral [1] with respect to this structure. For each $a \in K$, let $\bar{a}$ be the function in $\mathcal{F}$ such that $\bar{a}(x)=a$ for each $x \in M$. Then the operator "$-$" may be considered as an isomorphism of the topological additive group K into $\mathcal{F}$. Let us denote by $\bar{K}$ the image of K by this isomorphism. The topological additive group K can be identified with the subgroup $\bar{K}$ of $\mathcal{F}$ by this isomorphism and it holds that $K \subset \mathcal{G}$.

Now let i be the map of $\mathcal{S} \times K$ into J such that $i(X, \bar{a}) = \mu(X) \cdot a$ for each $X \in \mathcal{S}$ and $a \in K$. Then this map i satisfies the following conditions:

1) $i(X, a+b) = i(X, a) + i(X, b)$,

2) $i(X+Y, a) = i(X, a) + i(Y, a)$ if $XY=0$,

for each $X, Y \in S$, and $a, b \in K$. Further $\mathcal{J}$ is an extension of i.

Conversely, when such a map i is given, how can we extend the map i to an abstract integral $\mathcal{J}$? We shall give an answer to this question in the present part of the paper.

2. Construction of an abstract integral.

Assumption 1. *Let $(\mathcal{S}, \mathcal{F}, J)$ be an abstract integral structure and K a subgroup of $\mathcal{F}$. Let i be a map of $\mathcal{S} \times K$ into J satisfying the conditions:*

1) $i(X, a+b) = i(X, a) + i(X, b)$,

2) $i(X+Y, a) = i(X, a) + i(Y, a)$ if $XY=0$,

for each $X, Y \in \mathcal{S}$, and $a, b \in K$. Denote by $\mathcal{G}_0$ the subgroup of $\mathcal{F}$ generated by $\mathcal{S}K = \{Xa \mid X \in \mathcal{S}$ and $a \in K\}$ and by $\mathcal{G}$ the $\mathcal{F}$-completion [1] *of $\mathcal{G}_0$.*

Proposition 1. $\mathcal{G}_0 = \{\sum_{i=1}^{n} X_i a_i \mid X_i \in \mathcal{S}$ *and* $a_i \in K,\ i=1, 2, \cdots, n\}$

$= \{\sum_{i=1}^{n} X_i a_i \mid X_i \in \mathcal{S}$ *and* $a_i \in K,\ i=1, 2, \cdots, n,$ *and* $X_j X_k = 0\ (j \neq k)\}$.

Proof. It suffices to show that, for any $g = \sum_{i=1}^{n} X_i a_i \in \mathcal{G}_0$, where $X_i \in \mathcal{S}$ and $a_i \in K$, $i=1, 2, \cdots, n$, there exist $Y_j \in \mathcal{S}$ and $b_j \in K$,