100. On a Closed Graph Theorem for Topological Groups

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Let $\mathcal{C}$ be a class of Hausdorff topological groups satisfying the following: If for any Hausdorff topological group H there is a continuous almost open homomorphism of some member $G\in\mathcal{C}$ onto H then $H\in\mathcal{C}$. Let $E\in\mathcal{C}$ and let F be a $B_r(\mathcal{C})$-group. Then each almost open almost continuous homomorphism of E onto F, the graph of which is closed in $E\times F$, is continuous.

This theorem is stated for "into" homomorphisms in [2] p. 94, Theorem 5. The proof given there requires ontoness, however. The purpose of this note is to extend this theorem for "into" homomorphisms. But then one requires, in addition, that the range group F is abelian. When both E and F are abelian, a special case of the above theorem can be proved very easily as has been done by Baker [1]. Baker's method of proof requires that E and F both are abelian. In this note, we weaken Baker's assumption by assuming that only F is abelian. The proof of this extension is a slight modification of the proof given in my book ([2], p. 94, Theorem 5).

We shall follow the notations and terminology of [2] without any specific mention.

We prove the following

Theorem. *Let $\mathcal{C}$ be a class of Hausdorff topological groups satisfying the following: For any arbitrary Hausdorff topological group H if there exists a continuous almost open homomorphism of some member in $\mathcal{C}$ into H, then H is also in $\mathcal{C}$. Let $E\in\mathcal{C}$ and F an abelian $B_r(\mathcal{C})$-group. Let f be an almost continuous almost open homomorphism of E into F, the graph of which is closed in $E\times F$. Then f is continuous.*

Proof. Let u denote the initial topology on F. Let v denote the topology defined by the sets

$$U^*=\overline{f(\overline{f^{-1}(U)})},$$

where U runs over a fundamental system of neighborhoods of the identity of F, as in Theorem 5 ([2], p. 94). It is shown there that v is a Hausdorff topology such that F_v is a topological group. (Observe that "ontoness" of f is needed to show that $U^*\supset aU_1^*a^{-1}$. However,