178. *On the Minimality of the Polar Decomposition in Finite Factors*

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1. Ky Fan and A. J. Hoffman [2] established the following matrix inequalities: For every unitarily invariant norm of matrices,

(i) If A is an $n\times n$ matrix and $A=UH$ where U is unitary and H is positive-definite, then

$$\|A-U\|\leqq\|A-W\|\leqq\|A+U\|,$$

for every unitary matrix W [2; Theorem 1],

(ii) If A is an $n\times n$ matrix, then

$$\left\|A-\frac{A+A^*}{2}\right\|\leqq\|A-H\|,$$

for every hermitean matrix H [2; Theorem 2],

(iii) If H and K are hermitean $n\times n$ matrices, then

$$\|(H-i)(H+i)^{-1}-(K-i)(K+i)^{-1}\|\leqq 2\|H-K\|,$$

[2; Theorem 3].

In this note, we shall extend these inequalities of Fan and Hoffman for finite factors.

2. Throughout the note, let $\mathcal{A}$ be a finite factor with the (normalized) faithful normal trace φ such that $\varphi(1)=1$ (cf. [1]). For each $T\in\mathcal{A}$,

$$\|T\|_2^2=\varphi(T^*T)$$

defines a norm on $\mathcal{A}$, by which $\mathcal{A}$ becomes a prehilbert space. In a finite factor $\mathcal{A}$, if $T=V|T|$ is the polar decomposition of T, then the partially isometric operator V can be extended to a unitary $U\in\mathcal{A}$ such that $T=U|T|$.

3. We shall show that the unitary operator U appeared in the polar decomposition is one of the nearest unitary operators to the given T in $\mathcal{A}$, which will give an illustration of the polar decomposition in the finite factor $\mathcal{A}$:

Theorem 1. *Let T be any operator in $\mathcal{A}$ and $T=UH$ the polar decomposition of T, where U is a unitary, then for any unitary operator V in $\mathcal{A}$,*

$$\|T-U\|_2\leqq\|T-V\|_2\leqq\|T+U\|_2. \tag{1}$$

Proof. By the definition of the norm,

$$\|T-U\|_2^2=\|UH-U\|_2^2=\varphi(H^2-2H+1),$$