# 177. *On Extension of Semifield Valued Linear Functionals*

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In their book [1], M. Antonovski, V. Boltjanski, and T. Sarymsakov introduced a new notion called topological semifield. The present author and S. Kasahara [2] obtained a theorem of Hahn-Banach type for a semifield valued functional. M. Kleiber and W. Pervin generalized our result in their paper [4].

In this note, we shall generalize a theorem by V. Klee [3].

Let $E$ be a real linear space, and $\mathfrak{T}$ a set of linear transformations of $E$ into $E$. Let $p$ be a semifield valued subadditive functional on $E$, i.e., $p(x+y)\ll p(x)+p(y)$, $p(\alpha x)=\alpha p(x)$ for $\alpha\geq 0$.

For each $x\in E$, put

$$q(x)=\inf\{p(x+\sum_{i=1}^{k}T_iz_i)\,|\,T_i\in\mathfrak{T},\ y_i\in E,\ k \text{ is any positive integer}\}.$$

Let $f(x)$ be a semifield valued linear functional defined on a linear subspace $E_f$ of $E$ satisfying $f(x)\ll q(x)$. Then

$$0=f(0)\ll q(0)\ll p(\sum_{i=1}^{k}T_iz_i)\ll p(-x)+p(x+\sum_{i=1}^{k}T_iz_i).$$

Hence $0\ll p(-x)+q(x)$, which shows that $q$ is well-defined on $E$. $q(x)$ is a semifield valued positive homogeneous functional. Let $x, y\in E$, then for any saturated neighborhood $U$ of 0 in the semifield $S$, we can take $V$, $T_i$, $z_i$ such that

$$p(x+\sum_{i=1}^{k}T_iz_i)\ll q(x)+V,$$
$$p(y+\sum_{i=k+1}^{n}T_iz_i)\ll q(y)+V,$$
$$V+V\subset U.$$

Therefore we have

$$\begin{aligned}q(x+y)&\ll p(x+\sum_{i=1}^{k}T_iz_i+y+\sum_{i=k+1}^{n}T_iz_i)\\&\ll p(x+\sum_{i=1}^{k}T_iz_i)+p(y+\sum_{i=k+1}^{n}T_iz_i)\\&\ll q(x)+q(y)+V+V\\&\ll q(x)+q(y)+U.\end{aligned}$$

Hence we have $q(x+y)\ll q(x)+q(y)$. By the Hahn-Banach type extension theorem for a semifield valued functional (see K. Iséki and S. Kasahara [1]), we can find a linear functional $F$ satisfying $F=f$ on $E_f$, and $F\ll q$ on $E$. Then

$$F(Tz)\ll p(Tz+T(-z))=p(0)=0.$$