172. Semigroups Satisfying $xy^m = yx^m = (xy^m)^n$

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(Comm. by Kenjiro SHODA, M. J. A., Oct. 12, 1968)

Recently E. J. Tully [5] determined the semigroups satisfying an identity of the form $xy = y^m x^n$; Tamura [4], one of the authors, studied the semigroups satisfying an identity $xy = y^{m_1}x^{n_1} \cdots y^{m_k}x^{n_k}$; and Mead [2], the other author, found a necessary and sufficient condition in order that an implication, $x^n y^m = y^k x^l \to x^n y^m = y^n x^m$, hold in all semigroups. Related to these works the purpose of this paper is to find the structure of semigroups satisfying an identity of the form

$$(*) \qquad xy^m = yx^m = (xy^m)^n, \qquad n > 1.$$

Let L be a semilattice and $\{S_\alpha : \alpha \in L\}$ be a family of disjoint semigroups. If a semigroup S is a union of disjoint subsemigroups S'_α, $\alpha \in L$, and if S'_α is isomorphic with S_α for all α and $S'_\alpha S'_\beta \subseteq S'_{\alpha\beta}$ for all $\alpha, \beta \in L$, then S is called a semilattice-union of S_α, $\alpha \in L$, or a semilattice of S_α, $\alpha \in L$. A semigroup S is called a Clifford semigroup if S is a union of groups.

Lemma. *A Clifford semigroup S is commutative if and only if S is a semilattice-union of abelian groups.*

Proof. S is a semilattice-union of completely simple semigroups S_α by Theorem 4.6 [1]. Since S is commutative, each S_α is an abelian group. The converse is obtained from Theorem 4.11 [1].

Let I be an ideal of a semigroup S and $S/I \cong Z$. Then S is called an ideal extension of I by Z.

Theorem. *The following three statements are equivalent.*

(1) *A semigroup S satisfies the identity* $(*)$.

(2) *A semigroup S contains a commutative Clifford subsemigroup M and satisfies*

(2.1) $x^{k+1} = x$ *for all $x \in M$, where k is the greatest common divisor of $m-1$ and $n-1$.*

(2.2) $xy^m \in M$ *for all $x, y \in S$.*

(3) *A semigroup S is a semilattice-union of semigroups S_α, $\alpha \in L$, such that each S_α is an ideal extension of a group G_α by Z_α and the following conditions are satisfied:*

(3.1) *Each G_α is abelian and satisfies $x^k = e$ for all $x \in G_\alpha$, where e is the identity element of G_α, k being defined in* (2.1).