168. A Three Series Theorem

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1. We suppose throughout this paper that (m_n) tends to zero monotonically.

J. Meder [1] (cf. S. Kaczmarz [2]) has proved the following

Theorem I. *Denote by l_n, L_n, and $\tilde{L}_n$ the first logarithmic means of the three series*

$$(1)\qquad \sum_{n=1}^{\infty} a_n,\quad \sum_{n=1}^{\infty} a_n m_n,\quad \text{and}\quad \sum_{n=1}^{\infty} t_n\cdot\Delta^2 m_n$$

respectively, where $t_n=s_1+s_2+\cdots+s_n$ and $s_n=a_1+a_2+\cdots+a_n$. If

$$l_n=o(1/m_n)\qquad \text{as}\qquad n\to\infty$$

and

$$\Delta m_n=O(m_n/n\log n)\qquad \text{as}\qquad n\to\infty,$$

then $L_n=\tilde{L}_n+o(1)$ as $n\to\infty$.

He raized the problem ([1] P 471) whether this theorem holds also without any additional restriction or not and the problem ([1] P 472) to generalize this theorem by proving it e.g. in the case of weighted means or in the case of the Nörlund method of summation.

Let $p_n\geqq 0$, $p_1>0$, and $P_n=p_1+p_2+\cdots+p_n\to\infty$ as $n\to\infty$. The weighted mean w_n of the first series of (1) is defined by

$$w_n=(p_1s_1+p_2s_2+\cdots+p_ns_n)/P_n.$$

Similarly we denote by W_n and $\tilde{W}_n$ the weighted means of the second and the third series of (1).

The case $p_n=1/n$ is the first logarithmic mean. About the weighted means J. Meder and Z. Zdrojewski [3] proved the following

Theorem II. *Suppose that $p_n>0$, (p_n) is convex or concave and*

$$(2)\qquad 0<\liminf_{n\to\infty}(n+1)p_n/P_n\leqq\limsup_{n\to\infty}(n+1)p_n/P_n<\infty.$$

If

$$(3)\qquad w_n=o(m_n^{-1})\qquad \text{as}\qquad n\to\infty$$

and

$$(4)\qquad \Delta m_n=O(m_n/n)\qquad \text{as}\qquad n\to\infty,$$

then $W_n=\tilde{W}_n+o(1)$ as $n\to\infty$.

This theorem does not contain Theorem I as a particular case, since the first logarithmic means do not satisfy the condition (2). We shall prove the following