196. Note on Homogeneous Homomorphisms

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A homomorphism φ of a semigroup D onto a semigroup D' is called homogeneous if each congruence class of D induced by φ has a same cardinal number. A homogeneous homomorphism will be called h-homomorphism.

Let S be a set and T be a semigroup. Consider a mapping Θ of $T\times T$ into the set of all binary operations defined on S, $(\alpha, \beta)\Theta=\theta_{\alpha,\beta}$, $(\alpha, \beta)\in T\times T$, such that

$$(x\theta_{\alpha,\beta}a)\theta_{\alpha\beta,\gamma}y=x\theta_{\alpha,\beta\gamma}(a\theta_{\beta,\gamma}y)$$

for all $\alpha, \beta, \gamma\in T$, all $a\in S$.

Let $S\times T=\{(x, \alpha)\,;\ x\in S, \alpha\in T\}$. Given S, T, Θ, a binary operation is defined on $S\times T$ as follows:

(1) $$(x, \alpha)(y, \beta)=(x\theta_{\alpha,\beta}y, \alpha\beta).$$

Then $S\times T$ is a semigroup with respect to (1). The semigroup is called a general product of a set S by a semigroup T with respect to Θ and it is denoted by $S\overline{\times}_{\Theta}T$ or $S\overline{\times}T$. If a semigroup D is isomorphic onto some $S\overline{\times}_{\Theta}T$, $|S|>1$, $|T|>1$, then D is called general-product decomposable (gp-decomposable).

Theorem 1. *The following are equivalent:*

(2) *A semigroup D has a proper h-homomorphism.*

(3) *A semigroup D is gp-decomposable.*

(4) *There is a congruence ρ on D and there is an equivalence σ on D such that*

$$\rho\neq\omega,\quad \sigma\neq\omega,\quad \rho\cdot\sigma=\omega,\quad \rho\cap\sigma=\tau.$$

In Theorem 1, $\omega=D\times D$, $\tau=\{(x, x)\,;\ x\in D\}$ and $\rho\cdot\sigma=\{(x, y)\,;\ (x, z)\in\rho$ and $(z, y)\in\sigma$ for some $z\in D\}$.

The following theorem is concerned with the relationship between h-homomorphisms and homomorphisms.

Theorem 2. *If a semigroup D is homomorphic onto a semigroup T, there is a semigroup $\bar{D}$ such that*

(5) *D can be embedded into $\bar{D}$.*

(6) *$\bar{D}$ is h-homomorphic onto T and the homomorphism $\bar{D}\to T$ is the extension of the homomorphism $D\to T$.*

(7) *$\bar{D}\backslash D$ is an ideal of $\bar{D}$.*

Also there is a semigroup $\bar{D}_1$ such that $\bar{D}_1$ satisfies (5), (6), *and* (7′) *below:*