194. On Free Contents

By Takayuki TAMURA
University of California, Davis, California, U.S.A.

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1. Introduction. An S-indecomposable semigroup is a semigroup which has no semilattice-homomorphic image except a trivial one. We will call an S-indecomposable semigroup $\mathfrak{P}$-simple in the sense that a semigroup S is S-indecomposable if and only if it has no prime ideal, that is, S has no ideal I such that $I \neq S$ and $S \setminus I$ is a subsemigroup of S (cf. [1]).

Let S be a semigroup. Let $a_1, \cdots, a_n$ be a finite number of elements of S. All the elements x of S each of which is the product of all of $a_1, \cdots, a_n$ (admitting repeated use) form a subsemigroup of S. It is denoted by $C_S(a_1, \cdots, a_n)$ or C_S and is called the content of $a_1, \cdots, a_n$ is S. We notice that $a_1, \cdots, a_n$ need not be distinct. For example, however, $C_S(a)$ is different from $C_S(a, a)$ in general: $C_S(a) = \{a^i; i \geqq 1\}$ but $C_S(a, a) = \{a^i; i \geqq 2\}$. Let F_n be the free semigroup generated by $a_1, \cdots, a_n$. Then $C_{F_n}(a_1, \cdots, a_n)$ is called the free content of $a_1, \cdots, a_n$. The author did not use the terminology "content" and "$\mathfrak{P}$-simplicity" in the preceding papers [2], [3] but he proved there

(1) A free content is $\mathfrak{P}$-simple.

(2) A content is $\mathfrak{P}$-simple.

(3) A semigroup is a semilattice-union of $\mathfrak{P}$-simple semigroups.

(4) In the greatest semilattice-decomposition (S-decomposition) of a semigroup, each congruence class is $\mathfrak{P}$-simple.

The author discussed these in the two ways: one way is along the direction, (4)$\to$(3)$\to$(1)$\to$(2) after directly proving (4) [2]. The other way is along the direction, (1)$\to$(2)$\to$(4)$\to$(3) after directly proving (1) [3]. The concept of content is important and interesting but its structure has not been studied so much. In this short note we report a few results on free contents. The detailed proof will be published elsewhere [4].

2. Rank. The positive number n of $C_{F_n}(a_1, \cdots, a_n)$ is called the rank of a free content C_{F_n}. For simplicity the free content of rank n is denoted by $\mathcal{F}_n$.

$$\mathcal{F}_n = C_{F_n}(a_1, \cdots, a_n).$$

The letters $a_1, \cdots, a_n$ are called the generators of $\mathcal{F}_n$, but they are not elements of $\mathcal{F}_n$. We have the following theorem.