233. *A New Characterization of Hausdorff k-Spaces*

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Throughout, we shall assume all topological spaces are Hausdorff. A function $f: X \to Y$ from a space X to a space Y is said to be *weakly-continuous* if and only if $f^{-1}(y)$ is closed in X for each y in Y.

Let $f: X \to Y$ be a function from a space X to a space Y. The following are two properties which a space X may or may not satisfy:

$P_1(X)$: f is weakly continuous;

$P_2(X)$: for any filter base ([2], p. 211) $\{F_\lambda \mid \lambda \in \Lambda\}$ of compact sets of X, we have $f(\bigcap_{\lambda \in \Lambda} F_\lambda) = \bigcap_{\lambda \in \Lambda} f(F_\lambda)$.

Theorem 1. *If X is any space, then $P_1(X)$ implies $P_2(X)$.*

The proof of this theorem is straightforward. To our surprise, we discovered first the following:

Theorem 2. *If X is a k-space, then $P_2(X)$ implies $P_1(X)$; and hence $P_1(X)$ and $P_2(X)$ are equivalent.*

Proof. See the "necessity" part of the proof for Theorem 3, below.

Trying, in vain, to weaken the hypothesis of Theorem 2, we obtain the following characterization of k-spaces.

Theorem 3. *$P_1(X)$ and $P_2(X)$ are equivalent if and only if X is a k-space.*

Proof. *Necessity.* According to a theorem of Cohen [1], (see also [2], p. 248), X is a k-space if and only if it is a quotient space of a locally compact space, say Z. Let $p: Z \to X$ denote the natural projection (=quotient map). Suppose, $P_1(X)$ is false, i.e., there exists an element y in Y such that $f^{-1}(y)$ is not closed in X, then $p^{-1}(f^{-1}(y))$ is not closed in Z. Hence, there exists an element z in $Cl(p^{-1}(f^{-1}(y))$ such that $f(p(z)) \neq y$. Since Z is locally compact (and Hausdorff), there is a filter base $\{E_\lambda \mid \lambda \in \Lambda\}$ of compact neighborhoods E_λ of z such that $\bigcap_{\lambda \in \Lambda} E_\lambda = \{z\}$. Let $F_\lambda = p(E_\lambda)$ for all $\lambda \in \Lambda$, then $\{F_\lambda : \lambda \in \Lambda\}$ is a filter base of compact subsets of X such that $\bigcap_{\lambda \in \Lambda} F_\lambda = \{p(z)\}$. Then we have $f(\bigcap_{\lambda \in \Lambda} F_\lambda) = f(p(z))$; but $\bigcap_{\lambda \in \Lambda} f(F_\lambda)$ contains the element y, which is not in $f(\bigcap_{\lambda \in \Lambda} F_\lambda)$. This shows $f(\bigcap_{\lambda \in \Lambda} F_\lambda) \neq \bigcap_{\lambda \in \Lambda} f(F_\lambda)$, which contradicts $P_2(X)$. Thus, $P_2(X)$ and $P_1(X)$ are equivalent by the preceding and by Theorem 1.