37. On Generalized (A)-integrals. I

By Kaoru YONEDA
University of Osaka Prefecture

(Comm. by Kinjirô KUNUGI, M. J. A., March 12, 1969)

1. Introduction. To consider conjugate functions E.C. Tichmarsh introduced, in [1], the (Q)-integral. We say that $f(x)$ is (Q)-integrable in $[a, b]$ when there exists $\lim_{n\to\infty}\int_a^b [f(x)]_n dx$ and it is finite, and the limit is denoted by $(Q)\int_a^b f(x)dx$. But the (Q)-integral does not possess the additive property of integral. A.N. Kolmogorov showed, in [2], that if (Q)-integrable functions $f_i(x)$ $(i=1, 2)$ satisfies the condition: $n \operatorname{mes}(x\,;|f_i(x)|\geqq n)=o(1)$ $(i=1,2)$, for any α_i $(i=1,2)$, $\sum_i \alpha_i f_i(x)$ is also (Q)-integrable and $(Q)\int_a^b \sum_i \alpha_i f_i(x)dx=\sum_i \alpha_i (Q)\int_a^b f_i(x)dx$. If a (Q)-integrable function $f(x)$ satisfies the above condition, we say that $f(x)$ is $(\mathbf{A})$-integrable in $[a, b]$, and give a value of the $(\mathbf{A})$-integral by that of the (Q)-integral. A Lebesgue integrable function is $(\mathbf{A})$-integrable and both integrals have the same value. But there exists a function which is not $(\mathbf{A})$-integrable, for example $g(x)=(-1)^n/x$ where $1/n+1<x\leqq 1/n$ $(n=1,2,\cdots)$ and $g(0)=0$. K. Kunugi has proposed in [3] the notion of the generalized $(E.R.)$-integral by which this $g(x)$ is integrable in $[0,1]$.

In this paper, we state a generalization of the $(\mathbf{A})$-integral.

2. The generalization of (A)-integral. In this paper, consider only real valued functions which are measurable and almost everywhere finite in $[0,1]$ and denote the set of these functions by $\mathfrak{M}[0,1]$. Let $\mathfrak{H}\equiv\{h_n(x)\}_{n=1,2\cdots}$ be a sequence of non-negative Lebesgue integrable functions tending to infinite almost everywhere in $[0,1]$.

Definition of the $(\mathbf{A},\mathfrak{H})$*-integral. We say that* $f(x)$ *of* $\mathfrak{M}[0,1]$ *is* $(\mathbf{A},\mathfrak{H})$*-integrable in* $[0,1]$ *if* $f(x)$ *satisfies following* $[a]$ *and* $[b]$:

$[a]$ $\displaystyle\int_{(x;|f(x)|\geqq \alpha h_n(x))} h_n(x)dx=o(1)$ *for any* $\alpha>0$,

$[b]$ $\displaystyle\lim_{n\to\infty}\int_0^1 [f(x)]_{h_n}dx$ *exists and is finite, where*

$[f(x)]_{h_n}=f(x)$ *for* $|f(x)|<h_n(x)$ *and* $=0$ *for* $|f(x)|\geqq h_n(x)$.

The value of the integral is given by this limit and we denote it by $(\mathbf{A},\mathfrak{H})\int_0^1 f(x)dx$.

Especially put $h_n(x)=n\,u(x)$, where $u(x)$ is positive and Lebesgue