80. *Notes on Generalized Commuting Properties of Skew Product Transformations*

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1. Introduction. Let (M, Σ, m) be a measure space where M is a set of elements, Σ a σ-field of measurable subsets of M, and m a countably additive measure on Σ. An invertible measure-preserving transformation T of the measure space (M, Σ, m) is a one-to-one mapping of M onto itself such that if $B \in \Sigma$ then TB and $T^{-1}B \in \Sigma$ with $m(TB)=m(T^{-1}B)=m(B)$. Let $\mathfrak{G}$ be the group of all invertible measure-preserving transformations of (M, Σ, m) with I denoting the identity transformation on M. Associated with $T \in \mathfrak{G}$ is a sequence $C_n(T)$, $n=0, 1, 2, \cdots$, of subfamilies of $\mathfrak{G}$ defined inductively as follows:

$$C_o(T)=\{S \in \mathfrak{G} \mid S=I \text{ a.e.}\},$$
$$C_n(T)=\{S \in \mathfrak{G} \mid STS^{-1}T^{-1} \in C_{n-1}(T)\}.$$

It is clear that $C_n(T) \subset C_{n+1}(T)$ for each n. If there exists an integer N such that $C_N(T)=C_{N+1}(T)$ then $C_n(T)=C_N(T)$ for all $n \geqq N$. R. L. Adler [1] called $C_n(T)$ the nth class of generalized T-commuting transformations and defined the generalized commuting order $N(T)$ of T as follows:

$$N(T)=\begin{cases} \min\{n \mid C_n(T)=C_{n+1}(T)\} \text{ if there exists an integer } N \text{ such} \\ \text{that } C_N(T)=C_{N+1}(T), \\ \infty \text{ if } C_n(T) \neq C_{n+1}(T) \text{ for each } n. \end{cases}$$

Let H be the two-dimensional torus, i.e., $H=K \times K$, where $K =\{\exp[2\pi it] \mid 0<t \leqq 1\}$, equipped with the normalized Haar measure λ and let $T_{\gamma,\mu}$ denote the invertible measure-preserving transformation on H which is defined by

$$T_{\gamma,\mu}: (x, y) \rightarrow (x\gamma, y \cdot x^{\mu})$$

where γ is an element of K such that $\gamma^n \neq 1$ for every $n \neq 0$ and μ a non-zero integer. In [1], Adler asserted and proved the fact that $N(T_{\gamma,\mu})=2$. However I could not follow his proof. In this paper we shall assert and prove that $N(T_{\gamma,\mu})=3$. The method of the proof depends upon Adler's idea in [1].

2. Preliminaries. Let X be a half open unit interval $(0, 1]$ equipped with the usual topology. Since X is homeomorphic to the circle group K by the mapping ρ of X onto K which is defined by $\rho(x) =\exp[2\pi ix]$, we may consider X as the circle group equipped with the normalized Haar measure. Let $H=X \times X$ be the topological product