127. *Surjectivity of Linear Mappings and Relations*

By Shouro KASAHARA
Kobe Uuiversity

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In [3], Pták has proved the following theorem, in which (1) is called the closed relation theorem and (2) the open mapping theorem.

Theorem A. *Let E be a Banach space, F a normed linear space, R a closed linear subspace of $E\times F$, T a continuous linear mapping of E into F, and let $0<\alpha<\beta$. Let U and V be the unit balls of E and F respectively.*

(1) *If the set $RU+\alpha V$ contains a translate of βV, then $RE=F$ and $(\beta-\alpha)V\subset RU$.*

(2) *If the set $T(U)+\alpha V$ contains a translate of βV, then $T(E)=F$ and $(\beta-\alpha)V\subset T(U)$, so that T is open.*

A theorem which is similar to the assertion (2) is obtained by McCord [2]:

Theorem B. *Suppose T is a continuous linear mapping of a Banach space E into a normed linear space F, for which there are positive real numbers α and β, $\beta<1$, such that the following holds. For each y in F of norm 1, there exists an x in E of norm $\leq\alpha$ such that $\|y-Tx\|\leq\beta$. Then for each y in F, there exists an x in E such that $y=Tx$ and $\|x\|<\alpha(1-\beta)^{-1}\|y\|$.*

Theorem A has been generalized by Baker [1]. In this paper we shall state other generalizations of Theorem A and a generalization of Theorem B.

Throughout this paper, vector spaces are over the real or the complex numbers. Let E and F be two vector spaces, A a subset of E, and R be a subset of $E\times F$. By $R(A)$ we denote the set of all $y\in F$ such that $(x, y)\in R$ for some $x\in A$; the set $R(\{x\})$, where $x\in E$, will be denoted by $R(x)$. $S(A)$ denotes the union of all λA with λ in the closed unit interval $[0, 1]$, and A is said to be *star-shaped* if $S(A)=A$.

The essential part of our results is included in the following

Lemma. *Let E and F be two topological vector spaces, and R be a closed vector subspace of $E\times F$. Let B_0 be a sequentially complete bounded star-shaped convex subset of E such that $R(B_0)\neq\varnothing$, and let B be a bounded subset of F. Then $B\subset R(B_0)+\alpha B$ implies $(1-\alpha)B\subset R(B_0)$ for every $\alpha\in[0, 1)=[0, 1]\backslash\{1\}$.*

Proof. It suffices to consider the case where $\alpha\neq 0$. Let y be an arbitrary element of B. Since $B\subset R(B_0)+\alpha B$, there are points $x_1\in B_0$