9. *On a Class of Hypoelliptic Differential Operators*

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§1. **Introduction.** Let $A(x, y; \xi)$ and $B(x, y; \eta)$ be uniformly elliptic polynomials[1] in $\xi \in R^\nu$ and in $\eta \in R^\mu$, respectively, with coefficients in $C^\infty(\Omega)$ and $g(x)$ be a real valued function in $C^\infty(\Omega)$, not depending on y, where Ω is an open set of $R^\nu_x \times R^\mu_y$. In this paper, we consider the hypoellipticity[2] of linear partial differential operators of the form

(1) $$P = A(x, y; D_x) + g(x)^2 B(x, y; D_y),$$

where $D_x = (D_{x_1}, \cdots, D_{x_\nu})$ with $D_{x_j} = -i\partial/\partial x_j$ and $D_y = (D_{y_1}, \cdots, D_{y_\mu})$ with $D_{y_k} = -i\partial/\partial y_k$ $(i = \sqrt{-1})$. It is well known that if $g(x)$ vanishes at no point of Ω operator (1) is hypoelliptic in Ω. Indeed, we can immediately see that it is formally hypoelliptic there. For operator (1) in which $g(x)$ may vanish, we can prove

Theorem. *Suppose in operator* (1) *that A and B are uniformly elliptic in Ω and the coefficients of A are not dependent on the variable y and that there exists a multi-index $\alpha = (\alpha_1, \cdots \alpha_\nu) \in N^\nu$*[3] *such that $D^\alpha_x g = D^{\alpha_1}_{x_1} \cdots D^{\alpha_\nu}_{x_\nu} g$ vanishes at no point of Ω. Then the differential operator P of form* (1) *is hypoelliptic in Ω.*

This is motivated by the result of Dr. T. Matsuzawa (unpublished) that the operators on the (x, y)-plane: $D^{2l}_x + x^{2k} D^{2m}_y$ $(l, m = 1, 2, \cdots;$ $k = 0, 1, \cdots)$ are hypoelliptic in the plane (see [4]). One of the most important keys to the proof of Theorem is the inequality (H) which is stated in §2 and is one of the inequalities proved by Hörmander [2].

In §2 we prepare some lemmas and propositions, with the aid of which the proof of Theorem will be accomplished in §3.

§2. **Preliminaries.** Throughout this section we assume that A, B and g have the same meaning as in Theorem and that the degrees of A and B are $2l$ and $2m$ $(l, m = 1, 2, \cdots)$, respectively. First define norm $|||\cdot|||$ and its dual norm $|||\cdot|||'$ by

$$|||u|||^2 = \|D^l_x u\|^2 + \|g D^m_y u\|^2 + \|u\|^2, \quad |||v|||' = \sup_{u \in C^\infty_0(\Omega)} \frac{|\langle v, u\rangle|}{|||u|||},$$

1) The $A(x, y; \xi)$ is called *uniformly elliptic* in ξ, if there exists a positive constant c such that $\mathrm{Re}\, A_0(x, y; \xi) \geq c|\xi|^{2l}$ for all $\xi \in R^\nu$ and all $(x, y) \in \Omega$ where $2l$ is the degree of A and A_0 denotes the leading part of A.

2) We say that P is *hypoelliptic* in Ω, if every $u \in \mathcal{D}'(\Omega)$ is infinitely differentiable in every open set where Pu is infinitely differentiable.

3) We denote by N the set of non-negative integers.