36. *On a Riemann Definition of the Stochastic Integral.* I

By Sigeyosi OGAWA
Department of Applied Mathematics and Physics,
Faculty of Engineering, Kyoto University

(Comm. by Kinjirô KUNUGI, M. J. A., Feb. 12, 1970)

§ 1. Introduction. Let $\{\beta_t(\omega)\,;\, t\in[0,T]\}$ be a one-dimensional R^1-valued Brownien motion process, and let N_t^s be the smallest σ-algebra generated by $\{\beta_\tau(\omega)\,;\, s\leqslant\tau\leqslant t\}$. Let S be the class of functions $f_t(\omega)$ on $[0,T]\times\Omega$ satisfying the following conditions.

S.1) $f_t(\omega)$ is $B_{[s,t]}\times N_t^s$-measurable for every $t\in[s,T]$, where $B_{[s,t]}$ is the Borel field on the interval $[s,t]$.

S.2) $M\left(\int_s^t f_\tau^2(\omega)\,d\tau\right)<+\infty$ for $0\leqslant s\leqslant t\leqslant T$,

where $M(\cdot)$ denotes the expectation.

We will call a family of partitions $\Delta^{(n)}$ "canonical" if $\max(t_{i+1}^{(n)}-t_i^{(n)})\cdot n$ tends to a constant as $n\to\infty$, where $\Delta^{(n)}=\{0=t_0^{(n)}<t_1^{(n)}<\cdots<t_n^{(n)}=T\}$. Let us consider a following Riemann sum of a function $f_t(\omega)$ which belongs to the class S.

$$S_n(f)(\omega)=\sum_{i=0}^{n} f_{t_{i+1}^{(n)}+k(t_{i+1}^{(n)}-t_i^{(n)})}(\omega)(\beta_{t_{i+1}^{(n)}}(\omega)-\beta_{t_i^{(n)}}(\omega)) \tag{1}$$

where $0\leqslant k\leqslant 1$.

Now our aim is to investigate conditions for the existence of the l.i.m. $S_n(f)(\omega)$. As for this problem, it is well known that if the interpolation ratio k is fixed to zero the limit of the series $S_n(f)(\omega)$ exists and equals to the Ito's stochastic integral $\int_0^T f_t(\omega)d^0\beta_t(\omega)$,*) while if the interporlation ratios are taken arbitrarily in each interval $(t_i^{(n)}, t_{i+1}^{(n)})$ it may fail to converge.

We will concern only with the series (1), where the ratios of interporlation are fixed to a certain constant $k(0\leqslant k\leqslant 1)$ through all the intervals $(t_i^{(n)}, t_{i+1}^{(n)})$. Now the difficulty of this problem lies in the fact that the each random variables $f_{t_i^{(n)}+k(t_{i+1}^{(n)}-t_i^{(n)})}(\omega)$ $(i=0,1,\cdots,n-1)$ are not independent of the corresponding increments $\beta_{t_{i+1}^{(n)}}(\omega)-\beta_{t_i^{(n)}}(\omega)$ $(i=0,1,\cdots,n-1)$. So it seems to be necessary to put on the functions $f_t(\omega)$ one more condition which describes the way of dependence of $f_t(\omega)$ on the process $\beta_t(\omega)$.

To express this condition we will introduce a notion of β-differentiability of the functions $f_t(\omega)$ in § 2. With the help of this

*) To distinguish the Ito's integral from the other types of integrals the notation $d^0\beta_t$ will be used.