29. *Note on Covariance Operators of Probability Measures on a Hilbert Space*

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1. Introduction. Let $(\Omega, \mathcal{A}, \mu)$ be a probability measure space, and let $(\mathfrak{H}, \mathcal{B})$ denote a measurable space where $\mathfrak{H}$ is a real separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and $\mathcal{B}$ is the σ-algebra of Borel subsets of $\mathfrak{H}$. Let $x(\omega)$ denote a $\mathfrak{H}$-valued random variable, that is $\{\omega : x(\omega) \in B\} \in \mathcal{A}$ for all $B \in \mathcal{B}$; and let ν_x denote the probability measure (or distribution) on $\mathfrak{H}$ induced by μ and x, that is $\nu_x = \mu \circ x^{-1}$, or $\nu_x(B) = \mu(x^{-1}(B))$ for all $B \in \mathcal{B}$. Let $\mathfrak{M}(\mathfrak{H})$ denote the space of all probability measures on $\mathfrak{H}$; and let $\nu \in \mathfrak{M}(\mathfrak{H})$ be such that $\varepsilon_\nu\{\|x\|^2\} = \int \|x\|^2 d\nu < \infty$. Then the *covariance operator* S of ν is defined by the equation

$$\langle Sg, g \rangle = \int_{\mathfrak{H}} \langle f, g \rangle^2 d\nu(f) \tag{1}$$

(cf. Grenander [1], Parthasarathy [4], Prokhorov [5]). A linear operator L in $\mathfrak{H}$ is said to be an *S-operator* if it is a positive, self-adjoint operator with finite-trace; hence L is compact. S-operators play a fundamental role in the study of probability theory in Hilbert spaces (cf. [2, 3, 6, 10]). We recall that the function

$$\hat{\nu}(g) = \exp\{-1/2 \langle Sg, g \rangle\}, \quad g \in \mathfrak{H}, \tag{2}$$

is the *characteristic functional* (or Fourier transform) of a probability measure on $\mathfrak{H}$ iff S is an S-operator. Also, if ν is the measure corresponding to $\hat{\nu}$, then $\varepsilon_\nu\{\|x\|^2\} < \infty$; and S is the covariance operator of ν. We also recall that a measure ν on $\mathfrak{H}$ is *normal* (or *Gaussian*) iff $\hat{\nu}$ is of the form

$$\hat{\nu}(g) = \exp\{i\langle g_0, g \rangle - 1/2 \langle Sg, g \rangle\}, \tag{3}$$

where g_0 is a fixed element in $\mathfrak{H}$ and S is an S-operator. The element g_0 is the expectation of ν, and S its covariance operator.

Let $L_2(\Omega, \mathcal{A}, \mu, \mathfrak{H}) = L_2(\Omega, \mathfrak{H})$ denote the space of $\mathfrak{H}$-valued random variables $x(\omega)$ such that $\varepsilon_\mu\{\|x\|^2\} < \infty$, with norm defined by

$$[x]_2 = (\varepsilon_\mu\{\|x\|^2\})^{1/2}. \tag{4}$$

For any finite sequences $\{\xi_i\} \subset L_2(\Omega, \mathcal{A}, \mu) = L_2(\Omega)$ and $\{f_i\} \subset \mathfrak{H}$, put

$$\sum_{i=1}^n \xi_i(\omega) \odot f_i = \sum_{i=1}^n \xi_i(\omega) f_i (\text{mod } \mu). \tag{5}$$

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