22. Projective R-Modules with Chain Conditions on R/J

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Introduction. Let R be a ring and J its Jacobson radical.

In [6], I. Kaplansky proved the useful theorem that, if R is local, i.e. R/J is a division ring, then any projective R-module is free. In [4], Y. Hinohara generalized this theorem by proving that, if R is commutative and semilocal and its space of prime ideals spec (R) is connected, then any projective R-module is free. H. Bass, in his recent book [2], defines a noncommutative ring R to be *semilocal* if R/J is Artinian. §1 of the present paper is devoted to proving the analogue of the above results for semilocal rings. The local and commutative cases are consequences of this result (§1. Theorem).

In the case that the projective module is nonfinitely generated, our proof depends upon the following theorem of Bass [1].

Theorem. *If R/J is left Noetherian, then any uniformly $\aleph$-big projective R-module is free.*

The paper by Bass appeared as a reference in several articles. In particular the result has been useful if R is the group ring of a finite group [7, Theorem 7] and in the study of topological spaces homotopically equivalent to finite dimensional complexes [8, discussion of Theorem E].

The proof of the theorem is short and very elegant in the case that the cardinal $\aleph$ is uncountable [1, Theorem 2.2]. But the countable case is much more involved. In §2 of the present paper we give an elementary proof of Bass' result which would avoid the second half of the argument in the establishment of [1, Theorem 3.1] involving the juggling of infinite matrices. The proof is a positive response to [1, beginning of §3] and is the cleanest.

§1. Following [1], we shall call a ring R *p-connected* if for each nonzero projective R-module P, the image $\tau(P)$ of the natural pairing: $\mathrm{Hom}_R(P, R)\otimes P\to R$ is all of R. Regardless, $\tau(P)$ is a two-sided ideal. Any local ring R is p-connected. This statement can be proved without using the theorem of Kaplansky except for two simple lemmas. We need only assume that R/J be a simple ring. For, let $P\neq(0)$ be a projective R-module. It is easy to show that $P=\tau(P)P$ [see 3, p. 132]. Suppose $\tau(P)$ is a proper ideal of R. Then, since R/J is simple, $\tau(P)$

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