21. *Endomorphism Rings of Modules over Orders in Artinian Rings*

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Recently Small [7] proved that if a ring R has a right Artinian (classical) right quotient ring, then so does the endomorphism ring of a finitely generated projective right R-module.

On the other hand, it has been shown by Hart [3] that if a ring R has a semi-simple Artinian (classical) two-sided quotient ring Q, so does the endomorphism ring of a finitely generated torsion free right R-module M. In this case M is not necessarily projective, but its quotient module $M\otimes_R Q$ is projective as a right Q-module. Therefore, in the case where Q is non-semi-simple, it is interesting to obtain a condition under which finitely generated torsion free modules have projective quotient modules. The next proposition of this paper gives such a condition.

Proposition 1. *If a ring R has a two-sided perfect two-sided quotient ring Q, then the following conditions on a finitely generated right R-module M are equivalent*:

(1) *M is R-torsion free* (*in the sense of Levy* [5]) *and $M\otimes_R Q$ is Q-projective.*

(2) *M is isomorphic to a direct summand of a right R-module K such that $\sum_{i=1}^{n}\oplus R^{(i)}\supseteq K\supseteq \sum_{i=1}^{n}\oplus I_i$, where $R^{(i)}$ is a copy of R and I_i is a right ideal of $R^{(i)}$ containing a regular element.*

In this paper this condition (2), without assuming that M is finitely generated, will be called condition (A).

Then, we obtain the next main theorem which generalizes the above results of Small [7, Corollary 2] and Hart [3, Theorem 2].

Theorem 1. *If R is a ring with a right* (*resp. two-sided*) *Artinian right* (*resp. two-sided*) *quotient ring Q, then the endomorphism ring* $\mathrm{End}_R(M)$ *of a right R-module M satisfying condition* (A) *has also a right* (*resp. two-sided*) *Artinian right* (*resp. two-sided*) *quotient ring isomorphic to* $\mathrm{End}_Q(M\otimes_R Q)=\mathrm{End}_R(M\otimes_R Q)$.

As an application of Theorem 1, we shall prove finally

Theorem 2. *In Theorem 1, if Q is quasi-Frobenius and M is faithful, then* $\mathrm{End}_R(M)$ *has a quasi-Frobenius quotient ring which is isomorphic to the R-endomorphism ring of the injective hull of M.*