66. *The Information Theoretic Proof of Kac's Theorem*

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1. Introduction.

The object of this paper is to give information theoretic proofs of the following two well known theorems.

Theorem 1. *Let X_1 and X_2 be independent random variables and let*

$$Y_1 = X_1 \cos\gamma + X_2 \sin\gamma \tag{1}$$

$$Y_2 = -X_1 \sin\gamma + X_2 \cos\gamma. \tag{2}$$

If Y_1 and Y_2 are independent of each other for sufficiently small neighbor of some γ. Then the variables X_1 and X_2 are normally distributed.

Theorem 2. *Let $F(x)$ be a distribution function with mean zero and variance one. If for any positive σ_1 and σ_2 there exists $\sigma>0$ satisfying the following relation*

$$F\left(\frac{x}{\sigma_1}\right) * F\left(\frac{x}{\sigma_2}\right) = F\left(\frac{x}{\sigma}\right). \tag{3}$$

Then $F(x)$ is normal distribution, where the notation $*$ denotes the convolution of distribution functions.

Theorem 1 was proved by M. Kac 1 in a general form. And Theorem 2 was first proved by G. Pólya. We must assume appropriate conditions, as our proof is based on the information measures of C. E. Shannon, R. A. Fisher and YU. V. Linnik 1.

2. Notations and Lemmas.

We consider one dimensional random variable X with continuous probability density $p(x)$ and satisfying the conditions

$$\sup p(x) < \infty,\ E(X) = \int_{-\infty}^{\infty} xp(x)dx = 0, \quad \text{and} \quad D(X) = \int_{-\infty}^{\infty} x^2 p(x)dx. \tag{4}$$

And we put

$$I(X) = H(X) - \frac{1}{2}\log D(X) \tag{5}$$

following YU. V. Linnik 1 where

$$H(X) = -\int_{-\infty}^{\infty} p(x)\log p(x)dx \tag{6}$$

introduced by C. Shannon.