55. A Note on Norms of Compression Operators on Function Spaces

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1. In what follows, let $(X, \|\cdot\|)$ be a *rearrangement invariant Banach function space*, i.e. a Banach space of Lebesgue integrable functions over a (finite or infinite) interval $(0, l)$ which satisfies the following conditions:

(1.1) $|g|\leq|f|$,[1] $f\in X$ *implies* $g\in X$ *and* $\|g\|\leq\|f\|$;

(1.2) $0\leq f_n\uparrow$, $\|f_n\|\leq M$, $n\geq 1$ *implies* $f=\bigcup_{n\geq 1} f_n\in X$ *and* $\|f\|=\sup_{n\geq 1}\|f_n\|$;

(1.3) *If* $0\leq f\in X$ *and* g *is equimeasurable with* f, *then* $g\in X$ *and* $\|f\|=\|g\|$.

From (1.2) it follows that the norm $\|\cdot\|$ on X is *semicontinuous*, i.e. $0\leq f_n\uparrow f, f_n, f\in X$ implies $\|f\|=\sup_{n\geq 1}\|f_n\|$. We denote by σ_a $(a>0)$ *the compression operator* on X:

(1.4) $$\sigma_a f=f_a,\quad f\in X,$$

where f_a is given by $f_a(x)=f(ax)$, if $ax\leq l$, and $f_a(x)=0$ otherwise. Since X is rearrangement invariant, the linear operators $\sigma_a, a>0$ are bounded, and $\|\sigma_a\|\leq 1$, if $a\geq 1$, and $1\leq\|\sigma_a\|\leq a^{-1}$, if $0<a<1$ [8]. The values of $\|\sigma_a\|, a>0$ play an important role to describe some interesting properties of the function space X concerning some interpolation properties for classes of linear operators [4, 8, 9], the Hardy Littlewood maximal functions [7], or the conjugate functions [1, 5].

Now we put for $a>0$ and $n\geq 1$

(1.5) $$\gamma_a^n=\sup\{\|\sigma_a f\|;\ f\in S_n, \|f\|=1\},$$

where S_n denotes the set of all positive simple functions with at most n-distinct nonzero values. Then we have for every $a>0$

$$\gamma_a^1\leq\gamma_a^2\leq\cdots\leq\|\sigma_a\|.$$

When X is an $\Lambda(\varphi)$-space or an $M(\varphi)$-space [2], $\gamma_a^1=\|\sigma_a\|$ holds; When X is an Orlicz space L_Φ we have $\gamma_a^2=\|\sigma_a\|$ [4]. Since $\|\cdot\|$ on X is semicontinuous, $\|\sigma_a\|=\sup_{n\geq 1}\gamma_a^n$ holds for every $a>0$. Now the following questions are naturally raised:

i) *For every* $a>0$, *is* $\|\sigma_a\|=\gamma_a^2$ *true?*; *For an arbitrary* X, *does there exist an* $n\geq 1$ *such that* $\|\sigma_a\|=\gamma_a^n$ *holds for each* $a>0$?

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1) $|f|$ denotes the function $|f(x)|, x\in(0, l)$. $f\leq g$ means that $f(x)\leq g(x)$ a. e. on $(0, l)$.