# 54. *Properties of Ergodic Affine Transformations of Locally Compact Groups. II*

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This is a continuation of the preceding paper [5]. The followings shall be proved below: (1) If $G$ is a locally compact non-discrete abelian group which has an ergodic affine transformation $S$ with respect to a Haar measure on $G$ then $G$ is compact. (2) An affine transformation $S$ of a locally compact abelian group $G$ which has a dense orbit in $G$ is ergodic with respect to a Haar measure on $G$.

**Theorem 1.** *If $G$ is a locally compact non-discrete abelian group which has an ergodic affine transformation $S(x)=a+T(x)$ with respect to a Haar measure $\mu$ on $G$ then $G$ is compact.*

**Proof.** Let $G_0$ be the connected component of the identity 0 of $G$. Since $T$ is *bi*-continuous by virtue of [5, Theorem 1], $G_0$ is invariant under $T$. Thus $S$ induces an affine transformation $S_1$ of $G/G_0$ in the following way

$$S_1(x+G_0)=a+T(x)+G_0 \qquad \text{for} \qquad x+G_0 \in G/G_0.$$

It is clear that $S_1$ is ergodic with respect to a Haar measure $\mu_1$ on $G/G_0$.

*Case* I. Let $G_0$ be not open in $G$. Then $G/G_0$ is a locally compact totally disconnected non-discrete abelian group which has an ergodic affine transformation with respect to a Haar measure on $G/G_0$. Hence $G/G_0$ is compact by [5, Theorem 3], from which it follows easily that $G$ is compactly generated. Thus the well-known structure theorem for a locally compact, compactly generated abelian group (see [1, Theorem (9.8)]) implies that $G$ is topologically isomorphic with $R^p \times Z^q \times F$ for some nonnegative integers $p$ and $q$ and some compact abelian group $F$, where $R$ is the real line and $Z$ is the additive group of integers. But in the present case $q=0$, i.e., $G$ is topologically isomorphic with $R^p \times F$. For if $q \neq 0$ then $G/(R^p \times F)=Z^q$ is not finite, which is impossible since $R^p \times F$ is an open subgroup of $G$. It is clear that $F$ is invariant under $T$. So the ergodic affine transformation $S(x)=a+T(x)$ of $G$ induces an affine transformation $S_2(x+F)=a+T(x)+F$ of $G/F=R^p$ which is ergodic with respect to a Haar measure on $G/F=R^p$. By [3, Theorem 4], $G/F=R^p$ is compact, therefore $G=F$, i.e., $G$ is compact.

*Case* II. Let $G_0$ be open in $G$. Then $G/G_0$ is a discrete abelian