51. *A Generalization of the Riesz-Schauder Theory*

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We prove the following:

Theorem. *Let S be an analytic space and let $s \to K(s)$ be an analytic map of S into the ring of compact operators on a Banach space X. Then those points s of S for which $I+K(s)$ are not invertible form an analytic set in S.*

This is a generalization of the following assertion, which is a part of the Riesz-Schauder theory.

Corollary 1. *The spectrum of a compact operator is discrete.*

Proof. We apply the theorem to $I+sK$ and find that those s for which $I+sK$ are non-invertible form an analytic set in the complex plane $\boldsymbol{C}$, namely, discrete set of points or $\boldsymbol{C}$ itself. Because $I+sK$ is invertible when $s=0$, the latter case does not occur.

In the same way we can prove the following proposition which has applications in scattering theory.

Corollary 2. *Let $K(s)$ be a family of compact operators depending analytically on a parameter s in an open subset U of the complex plane $\boldsymbol{C}$. Then the set of all s for which $I+K(s)$ are non-invertible is either equal to U itself, or discrete in U.*

Proof of the Theorem.

We use a method given by Donin [1].

Since the concept of analytic subset is local, it suffices to consider a neighborhood of a fixed point $s_0 \in S$. Let N_0 and R_0 be the kernel and the range, respectively, of the map $I+K(s_0): X \to X$. Since $K(s_0)$ is compact, N_0 is of finite dimension, R_0 is of finite co-dimension, and therefore both are topological direct summands.

Let $X=N_0 \oplus Y$ and let P_0 be a continuous projection to R_0. Then the map $Y(s)=P_0 \circ [I+K(s)]|_Y : Y \to R_0$ gives, for $s=s_0$, an isomorphism $Y \cong R_0$. Since $Y(s)$ is continuous in s, $Y(s)$ is invertible for s sufficiently close to s_0. So, we can construct a map $h(s): N_0 \oplus R_0 \to X$ which is defined by $h(s)(y,z)=\{I-Y(s)^{-1} \circ P_0 \circ (I+K(s))\}y+Y(s)^{-1}z$, where $(y,z) \in N_0 \oplus R_0$. When $s=s_0$, this is an isomorphism $N_0 \oplus R_0 \cong X$, so $h(s)$ is an isomorphism for any s in some neighborhood of s_0, and we have, for s sufficiently near s_0, $\dim \ker (I+K(s)) = \dim \ker \{(I+K(s)) \circ h(s)\}$. On the other hand, we can show that $\ker \{(I+K(s) \circ h(s)\} \subset N_0$. In fact, for $(y,z) \in N_0 \oplus R_0$,