78. *An Extremal Property of the Polar Decomposition in von Neumann Algebras*

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1. In this paper, we shall concern with a polar decomposition of an operator in a von Neumann algebra in a connection with an extreme point of the unit ball of the algebra. Substantially, we shall show that an operator of a von Neumann algebra is the product of an extreme point of the unit ball and a positive operator in the algebra (Theorem 1).

As a few applications, we shall have a characterization of a finite von Neumann algebra and that every element of the unit ball of a von Neumann algebra is the average of two extreme points.

2. Let $\mathcal{H}$ be a Hilbert space. By an operator we shall mean a bounded linear operator acting on $\mathcal{H}$. For a C*-algebra $\mathcal{A}$ of operators, by $(\mathcal{A})_1$ we shall mean the *unit ball* of $\mathcal{A}$. An extreme point of $(\mathcal{A})_1$ will be called simply an *extreme point of* $\mathcal{A}$. Following after Halmos [5; p. 63] if U and V are partial isometries, write $U \leqq V$ in case V agrees with U on the initial space of U.

Let $\mathcal{L}(\mathcal{H})$ be the algebra of all operators on $\mathcal{H}$, then every element in $\mathcal{L}(\mathcal{H})$ is the product of a maximal partial isometry (with respect to the above partial order) and a positive operator [5; p. 69]. A maximal partial isometry is an isometry or a co-isometry [5; p. 64]. By Kadison [6], for a factor, a necessary and sufficient condition that a partial isometry be an extreme point of the unit ball is that the partial isometry be an isometry or a co-isometry. Therefore, every operator on $\mathcal{H}$ has a representation as the product of an extreme point of $\mathcal{L}(\mathcal{H})$ and a positive operator.

Furthermore, let $\mathcal{A}$ be a finite von Neumann algebra on $\mathcal{H}$. It is essentially known that any element in $\mathcal{A}$ is the product of a unitary elment and a positive element of $\mathcal{A}$, and in finite factors, this fact is used repeatedly (e.g. [1], [4]). In a finite von Neumann algebra, the set of all extreme points of the unit ball is that of all unitary operators (cf. [2], [7], [10]). Therefore, any element of $\mathcal{A}$ is the product of an extreme point and a positive element.

We shall show the above fact is also true for a general von Neumann algebra:

Theorem 1. *Let $\mathcal{A}$ be a von Neumann algebra. Then any ele-*