147. *Some Conditions on an Operator Implying Normality. III*

By S. K. BERBERIAN
The University of Texas at Austin

(Comm. by Kinjirô KUNUGI, M. J. A., Sept. 12, 1970)

The purpose of this note is to record some generalizations of results proved recently by I. Istrăţescu [9].

Notations. If T is an operator (bounded linear, in Hilbert space), we write $\sigma(T)$ for the spectrum of T, $\omega(T)$ for the Weyl spectrum of T, $W(T)$ for the numerical range of T and Cl $W(T)$ for its closure, and $\hat{T}$ for the image of T in the Calkin algebra (the algebra of all operators modulo the ideal of compact operators). We refer to [2]-[4] or [7] for terminology.

Theorem 1. *If T is a seminormal operator such that $T^p = ST^{*p}S^{-1} + C$, where p is a positive integer, C is compact, and $0 \notin \mathrm{Cl}\, W(S)$, then T is normal.*

Proof. By hypothesis, $\hat{T}^p = \hat{S}\hat{T}^{*p}\hat{S}^{-1}$; moreover, it is easy to see that $\bar{W}(\hat{S}) \subset \bar{W}(S) = \mathrm{Cl}\, W(S)$, where $\bar{W}$ denotes closed numerical range [5, Theorem 3], thus $0 \notin \bar{W}(\hat{S})$. By a theorem of J. P. Williams [12], $\sigma(\hat{T}^p)$ is real, i.e., $\{\lambda^p : \lambda \in \sigma(\hat{T})\}$ is real, thus $\sigma(\hat{T})$ lies entirely on p lines through the origin. Since $\partial\omega(T) \subset \sigma(\hat{T})$, where ∂ denotes boundary (this is true for any operator [cf. 6, Theorem 2.2]), it follows that $\omega(T)$ also lies on these lines, and in particular $\omega(T)$ has zero area. Since Weyl's theorem holds for T [1, Example 6], $\sigma(T) - \omega(T)$ is countable; thus $\sigma(T)$ also has zero area, therefore T is normal by a theorem of C. R. Putnam [11].

{The following argument is of interest because it uses far less than the full force of Putnam's deep theorem. Assuming T is a seminormal operator such that $\omega(T)$ lies on finitely many lines through (say) the origin, we assert that T is normal. We can suppose T hyponormal. Writing $T = T_1 \oplus T_2$ with T_1 normal and $\sigma(T_2) \subset \omega(T)$ [3, Corollary 6.2], we are reduced to the case that $\sigma(T)$ lies on finitely many lines through the origin. Assume to the contrary that T is nonnormal. Splitting off the maximal normal direct summand of T, we can suppose that T has no normal direct summands. In particular, $\sigma(T)$ can have no isolated points (these would be eigenvalues, with reducing eigenspaces). Rotating T by a scalar of absolute value 1, we can suppose that the positive real axis contains a point of $\sigma(T)$ of maximum modulus, say