71. *A Note on the Number of Generators of an Ideal*

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Through this note, we mean by a ring a commutative ring with identity 1. Let R be a noetherian ring and A be an ideal of R. O. Forster showed that, if AR_M is generated by at most r elements for any maximal ideal M of R, then A is generated by at most $r+\mathrm{Alt}.\, R$ elements, where Alt. R is the Krull dimension of R (cf. O. Forster [1]). In this note, we shall study the number of generators of an ideal and improve the above Forster's result, that is:

Theorem 1. *Let R be a ring and A be a finitely generated ideal of R. Assume that*: (1) *there are only a finite number of maximal ideals of R which contain A and* (2) *AR_M is generated by at most r elements for any maximal ideal M of R. Then A is generated by at most $r+1$ elements.*

Theorem 2. *Let R be a noetherian ring and A be an ideal of R such that* $\mathrm{Alt}.\, R/A<\infty$. *Assume AR_M is generated by at most r elements for any maximal ideal M of R. Then A is generated by at most* $r+\mathrm{Alt}.\, R/A+1$ *elements.*

To prove these theorems we need the following lemmas.

Lemma 1. *Let R be a ring. Assume $0=Q_1\cap\cdots\cap Q_n$ be an irredundant decomposition of zero ideal of R (not necessarily primary decomposition). If $Q_1+Q_j=R$ $(j=2,\cdots,n)$, then Q_1 is a principal ideal.*

Proof. Since $Q_1\oplus Q_2Q_3\cdots Q_n=R$, we can take $x\in Q_1$ and $y\in Q_2\cdots Q_n$ such that $x+y=1$. For any element $z\in Q_1$, $z=zx+zy=zx$, so we have $Q_1=xR$.

Lemma 2. *Let R be a ring and A be a finitely generated ideal which contains an ideal B. If $AR_M=BR_M$ for any maximal ideal M which contains A, then $A=B$ or $A=xR+B$ for some element x of A.*

Proof. Since A is finitely generated, $AR_M=BR_M$ implies $B:A\not\subset M$ for any maximal ideal M which contains A. So we have $(A\cap(B:A))R_M=BR_M$ for any maximal ideal M of R, hence $B=A\cap(B:A)$. If $B:A=R$ then $B=A$. If $B:A\neq R$ then $A+(B:A)=R$ since $B:A\not\subset M$ for any maximal ideal M which contains A. So Lemma 1 implies $A=B+xR$ for some $x\in A$ by considering R/B and A/B.

Lemma 3. *Let R be a ring and A be an ideal of R. Assume that*: (1) *there are only a finite number of maximal ideals $M_1,\cdots,M_n$ which contain A and* (2) *AR_{M_i} is generated by at most r elements for every i.*