65. On the Numerical Range of an Operator

By Takayuki FURUTA*) and Ritsuo NAKAMOTO**)

(Comm. by Kinjirô KUNUGI, M. J. A., March 12, 1971)

1. Introduction. In this paper, an *operator* T means a bounded linear operator acting on a complex Hilbert space H.

Following after Halmos [6] we define the *numerical range* $W(T)$ and the *numerical radius* $w(T)$ of T as follows:

$$W(T)=\{(Tx, x)\,;\,\|x\|=1\}$$

and

$$w(T)=\sup\{|\lambda|\,;\,\lambda\in W(T)\}.$$

$W(T)$ is convex and the closure $\overline{W(T)}$ of $W(T)$ contains the *spectrum* $\sigma(T)$ of T; $w(T)$ is a norm equivalent to the operator norm $\|T\|$ which satisfies

$$\frac{1}{2}\|T\|\leqq w(T)\leqq\|T\|$$

and the *power inequality* ([3]):

$$w(T^n)\leqq w(T)^n \qquad (n=1,2,\cdots).$$

Definition 1 ([6]). An operator T is said to be *convexoid* if

$$\overline{W(T)}=\mathrm{co}\,\sigma(T),$$

where $\mathrm{co}\,\sigma(T)$ means the convex hull of the spectrum $\sigma(T)$ of T.

Definition 2 ([6]). An operator T is said to be *spectraloid* if

$$w(T)=r(T),$$

where $r(T)$ means the *spectral radius* of T:

$$r(T)=\sup\{|\lambda|\,;\,\lambda\in\sigma(T)\}.$$

By [4], it is known that T is a spectraloid if and only if

$$w(T)^n=w(T^n) \qquad (n=1,2,\cdots).$$

Definition 3 ([6]). An operator T is said to be *normaloid* if

$$\|T\|=r(T),$$

or equivalently

$$\|T\|^n=\|T^n\| \qquad (n=1,2,\cdots).$$

The classes of normaloids and convexoids are both contained in the class of spectraloids (cf. [6; p. 115]).

Definition 4 ([1]). A unitary operator U is said to be *cramped* if $\sigma(U)$ is contained in some semicircle:

$$\sigma(U)\subset\{e^{i\theta}\,;\,\theta_1\leqq\theta\leqq\theta_2,\,\theta_2-\theta_1<\pi\}.$$

Let $\boldsymbol{B}(H)$ be the algebra of all bounded linear operators acting on

*) Faculty of Engineering, Ibaraki University, Hitachi.
**) Tennoji Senior Highschool, Osaka.